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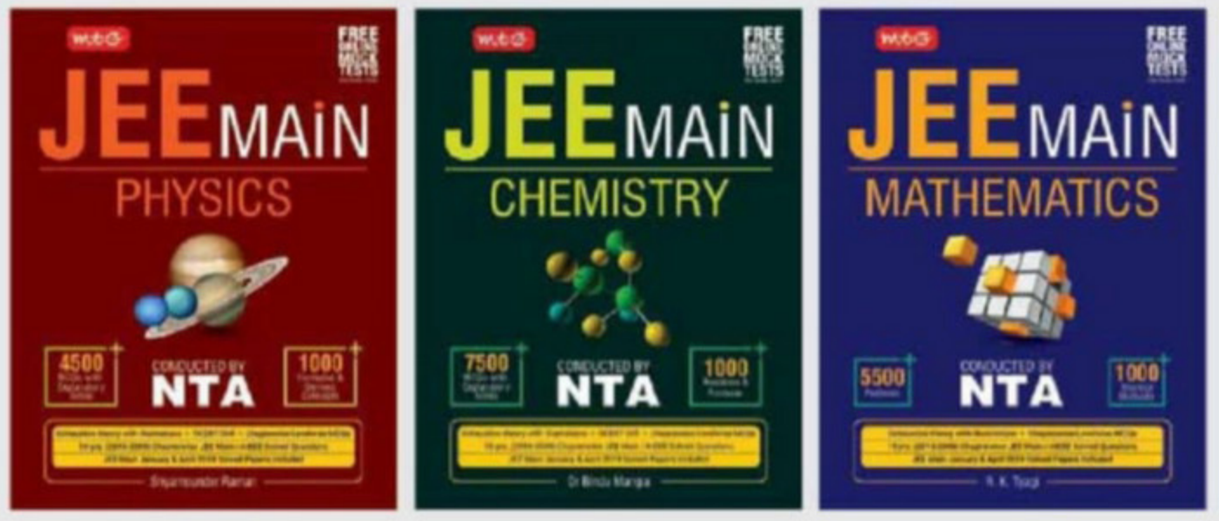

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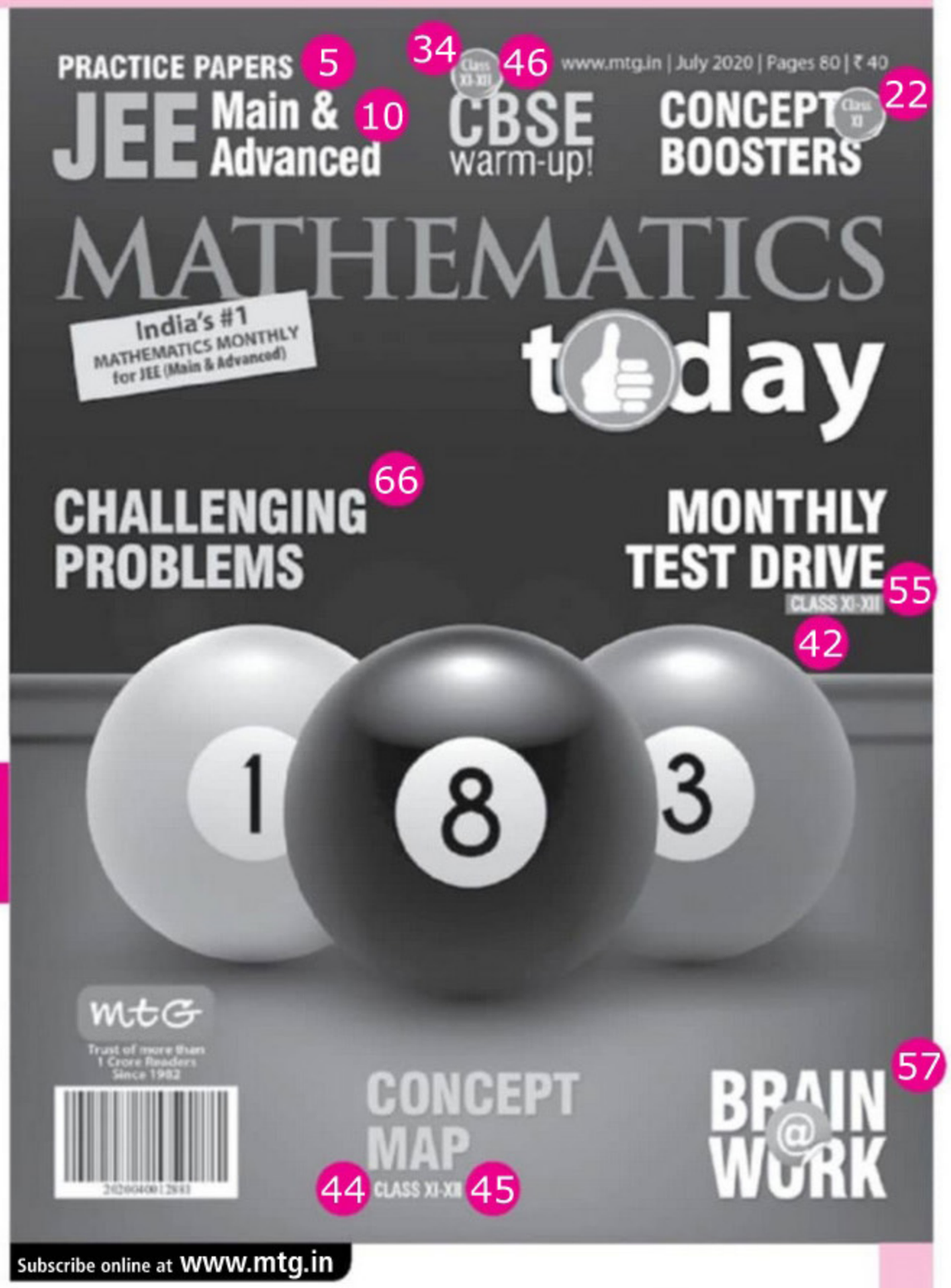

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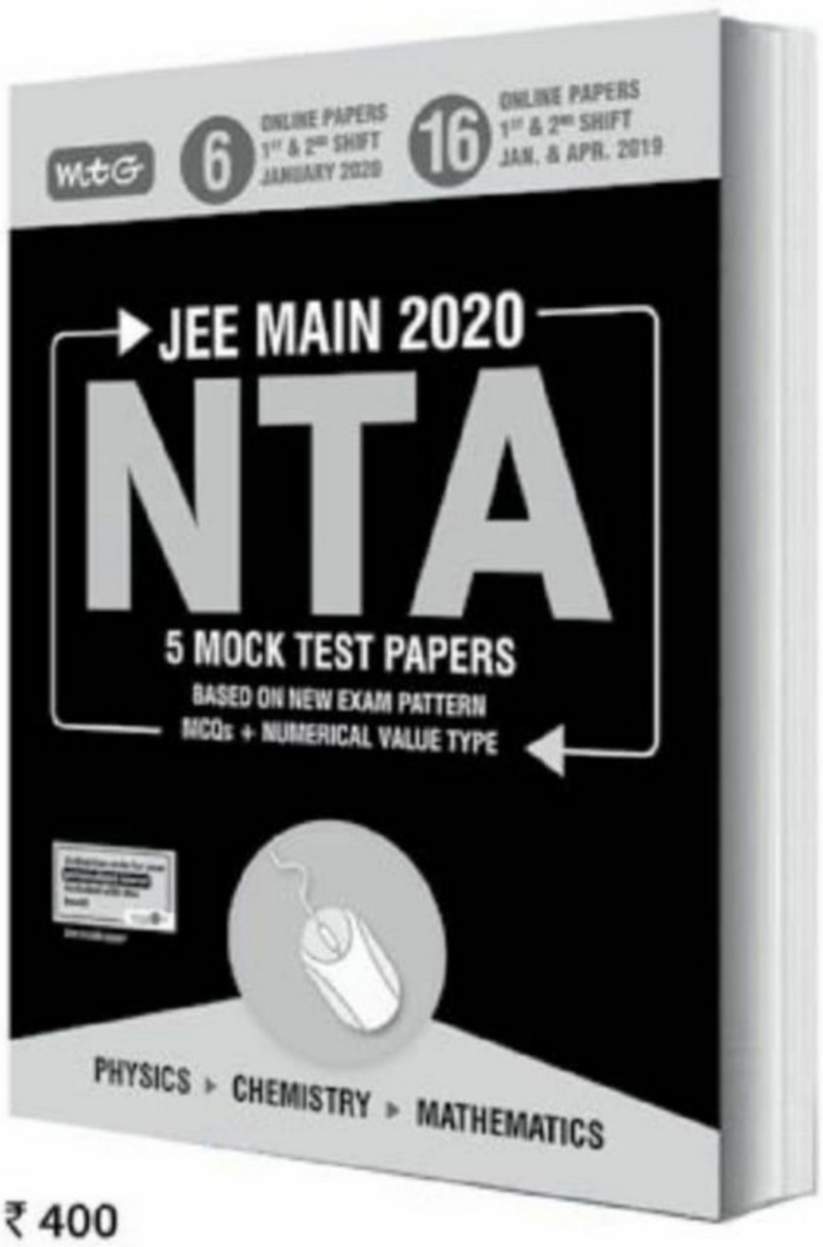

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GEAR UP FOR JEE MAIN

EXAM ON
1st - 6th September
2020

with Numerical Value Type Questions

1. Let W denote the words in the English dictionary. Define the relation R by :
$R = \{(x, y) \in W \times W |$ the words x and y have at least one letter in common$\}$. Then R is
(a) not reflexive, symmetric and transitive.
(b) reflexive, not symmetric and not transitive.
(c) not reflexive, not symmetric and transitive.
(d) none of these

2. If $z^2 + z + 1 = 0$, where z is a complex number, then the value of
$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+....+\left(z^6+\frac{1}{z^6}\right)^2$ is
(a) 18 (b) 54 (c) 6 (d) 12

3. f is defined in $[-5, 5]$ as
$f(x)=\begin{cases} x, & \text{if } x \in Q \text{ (rational)} \\ -x, & \text{if } x \in S \text{ (irrational)} \end{cases}$
then
(a) $f(x)$ is continuous at every x except $x = 0$.
(b) $f(x)$ is discontinuous at every x except $x = 0$.
(c) $f(x)$ is continuous everywhere.
(d) $f(x)$ is discontinuous everywhere.

4. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x. The expression $3b^2x^2 + 6bcx + 2c^2$ is
(a) less than $4ab$ (b) greater than $-4ab$
(c) less than $-4ab$ (d) greater than $4ab$

5. If the coefficient of x^7 in $\left[ax^2+\left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax-\left(\frac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation
(a) $a + b = 1$ (b) $a - b = 1$
(c) $ab = 1$ (d) $\frac{a}{b} = 1$

6. Let f be the differentiable for $\forall\ x$. If $f(1) = -2$ and $f'(x) \geq 2$ for $[1, 6]$, then
(a) $f(6) < 8$ (b) $f(6) \geq 8$
(c) $f(6) = 5$ (d) $f(6) < 5$

7. Consider the statements
(i) Sun rises or moon sets.
(ii) All integers are positive or negative.
(iii) Two lines intersect at a point or are parallel.
(iv) The school is closed if it is holiday or a Sunday.
Which of the above statements is "inclusive or" statement ?
(a) (i) and (ii) (b) (iv)
(c) (i) and (iii) (d) none of these

8. If $f(x)=\int_0^x (t^2+2t+2)dt, 2 \leq x \leq 4$, then
(a) the maximum value of $f(x)$ is $\frac{136}{3}$.
(b) the minimum value of $f(x)$ is 10.
(c) the maximum value of $f(x)$ is 26.
(d) none of these

9. $\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3+\cos^2(x+3\pi)]dx$ is equal to
(a) $\frac{\pi^4}{32}$ (b) $\frac{\pi^4}{32}+\frac{\pi}{2}$
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{2}-1$

10. Let a, b and c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is
(a) the arithmetic mean of a and b.
(b) the geometric mean of a and b.
(c) the harmonic mean of a and b.
(d) equal to zero.

11. The line passes through the points $(5, 1, a)$ and $(3, b, 1)$ crosses the yz-plane at the point $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$, then

(a) $a = 4, b = 6$ (b) $a = 6, b = 4$
(c) $a = 8, b = 2$ (d) $a = 2, b = 8$

12. The area bounded by the curves $y = ax^2$ and $x = ay^2$ is equal to 1. Then $a =$

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{3}$

13. Evaluate : $\int \frac{x\cos x+1}{\sqrt{2x^3e^{\sin x}+x^2}}dx$

(a) $\ln(2xe^{\sin x}+1)+c$
(b) $\ln(2xe^{\sin x}-1)+c$
(c) $\ln\left(\frac{2xe^{\sin x}+1}{2xe^{\sin x}-1}\right)+c$
(d) $\ln\left(\frac{\sqrt{2xe^{\sin x}+1}-1}{\sqrt{2xe^{\sin x}+1}+1}\right)+c$

14. The points (0, 8/3), (1, 3) and (82, 30)
(a) form an obtuse angled triangle.
(b) form an acute angled triangle.
(c) form a right angled triangle.
(d) lie on a straight line.

15. The differential equation of family of circles with fixed radius 5 units and centre lies on the line $y = 2$, is
(a) $(y-2)(y')^2 = 25-(y-2)^2$
(b) $(y-2)^2(y')^2 = 25-(y-2)^2$
(c) $(x-2)(y')^2 = 25-(y-2)^2$
(d) $(x-2)^2(y')^2 = 25-(y-2)^2$

16. Let X be a set containing 10 elements and $P(X)$ be its power set. If A and B are picked up at random from $P(X)$, with replacement, then the probability that A and B have equal number of elements, is

(a) $\frac{{}^{20}C_{10}}{2^{10}}$ (b) $\frac{(2^{10}-1)}{2^{20}}$
(c) $\frac{(2^{10}-1)}{2^{10}}$ (d) $\frac{{}^{20}C_{10}}{2^{20}}$

17. If the chord $y = mx + 1$ of circle $x^2 + y^2 = 1$ subtends an angle of measure 45° at the major segment of the circle, then value of m is
(a) $2\pm\sqrt{2}$ (b) $-2\pm\sqrt{2}$
(c) $-1\pm\sqrt{2}$ (d) none of these

18. The function $f:\left[-\frac{1}{2},\frac{1}{2}\right]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ defined by $f(x)=\sin^{-1}(3x-4x^3)$ is
(a) both one-one and onto
(b) onto but not one-one
(c) one-one but not onto
(d) neither one-one nor onto

19. A straight line is normal to both the parabolas $y^2 = x$ and $x^2 = y$. The distance of the origin from it is

(a) $\frac{3}{4\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{3\sqrt{2}}$

20. A vector $\vec{r}$ is equally inclined with the coordinate axes. If the tip of $\vec{r}$ is in the positive octant and $|\vec{r}|=6$, then $\vec{r}$ is

(a) $2\sqrt{3}(\hat{i}-\hat{j}+\hat{k})$ (b) $2\sqrt{3}(-\hat{i}+\hat{j}+\hat{k})$
(c) $2\sqrt{3}(\hat{i}+\hat{j}-\hat{k})$ (d) $2\sqrt{3}(\hat{i}+\hat{j}+\hat{k})$

NUMERICAL VALUE TYPE

21. If $a_1, a_2, a_3, \ldots, a_n \in$ G.P., then the value of the determinant

$$\Delta=\begin{vmatrix}\log a_n & \log a_{n+1} & \log a_{n+2}\\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5}\\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8}\end{vmatrix}$$ equals _____.

22. The number of distinct arrangements of the letters of the word IITJEE that can be made preserving the order in which the vowels (I, I, E, E) occur and not counting the original arrangement IIT-JEE is _____.

23. A person purchases one kilogram of tomatoes from each of the 4 places at the rate of 1 kg, 2 kg, 3 kg and 4 kg per rupee respectively. On the average he has purchased x kg of tomatoes per rupee, then the value of x is _____ kg per rupee.

24. If Rolle's theorem holds for the function $f(x) = 2x^3 + bx^2 + cx, x \in [-1, 1]$, at the point $x = 1/2$, then $|2b + c|$ equals _____.

25. The radius of a right circular cylinder increases at a constant rate $\frac{1}{3}\pi$ cm/s. Its altitude is a linear function of the radius and increases three times as fast as radius. When the radius is 1 cm, the altitude is 6 cm. When the radius is 3 cm, the volume is increasing at rate of n cm^3/s. The value of n (in cm^3/s) is equal to _____.

SOLUTIONS

1. (d)

2. (d) : $z^2 + z + 1 = 0 \implies z = \omega, \omega^2$

$$\therefore \quad \left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+.....+\left(z^6+\frac{1}{z^6}\right)^2$$

$$=(\omega+\omega^2)^2+(\omega+\omega^2)^2+\left(\omega^3+\frac{1}{\omega^3}\right)^2+(\omega+\omega^2)^2$$
$$+(\omega+\omega^2)^2+\left(\omega^3+\frac{1}{\omega^3}\right)^2$$

$$= 4(\omega+\omega^2)^2 + 2\left(\omega^3+\frac{1}{\omega^3}\right)^2 = 4(1) + 2(2^2)$$

$$= 4 + 8 = 12$$

3. (b)

4. (b) : The roots of $bx^2 + cx + a = 0$ are imaginary *i.e.*, $c^2 - 4ab < 0 \implies c^2 < 4ab$

Again, the coeff. of x^2 in $3b^2x^2 + 6bcx + 2c^2$ is +ve, so the minimum value of the expression

$$=-\left[\frac{36b^2c^2-4(3b^2)(2c^2)}{4(3b^2)}\right]=-\frac{12b^2c^2}{12b^2}=-c^2$$

As $c^2 < 4ab$, we have, $-c^2 > -4ab$

Thus, the minimum value is $-4ab$.

5. (c) : T_{r+1} of $\left(ax^2+\frac{1}{bx}\right)^{11} = {}^{11}C_r(ax^2)^r\left(\frac{1}{bx}\right)^{11-r}$

T_{r+1} of $\left(ax-\frac{1}{bx^2}\right)^{11} = {}^{11}C_r(ax)^r\left(-\frac{1}{bx^2}\right)^{11-r}$

$\therefore$ Coeff. of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11} = {}^{11}C_6\frac{a^6}{b^5}$

and coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11} = {}^{11}C_5\frac{a^5}{b^6}$

Now, ${}^{11}C_6\frac{a^6}{b^5} = {}^{11}C_5\frac{a^5}{b^6} \implies ab = 1$

6. (b) : Let if possible $f'(x) = 2$ for $[1, 6]$.

$\Rightarrow f(x) = 2x + c$ (Integrating both sides w.r.t. x)

$\therefore f(1) = 2 + c = -2$

$\Rightarrow c = -4$

$\therefore f(x) = 2x - 4$

$\therefore f(6) = 2 \times 6 - 4 = 8$

$\therefore f(6) \geq 8.$

7. (b) : (i) Sun rises or moon sets is an "exclusive or" statement as sun rises and moon sets during day time.

(ii) Since, all integers cannot be both positive as well as negative. Therefore, statement is "exclusive or".

(iii) It is not possible for two lines to intersect and parallel at the same time. So, this statement is "exclusive or" statement.

(iv) It is fact that school is closed on holiday as well as on Sunday. So, statement "The school is closed if it is a holiday or Sunday.

Hence, the statement (iv) is an "inclusive or" statement.

8. (a) : $f'(x) = x^2 + 2x + 2 = (x + 1)^2 + 1 > 0 \ \forall \ x$

$\therefore$ $f(x)$ is monotonically increasing in $[2, 4]$

$\therefore$ Min. $f(x) = f(2)$ and Max. $f(x) = f(4)$

$\therefore$ Min. $f(x)=\int_0^2(t^2+2t+2)dt=\frac{32}{3}$ and

$$\text{Max.} f(x)=\int_0^4(t^2+2t+2)dt=\frac{136}{3}.$$

9. (c): Let $I = \int_{-3\pi/2}^{-\pi/2}[(x+\pi)^3+\cos^2(x+3\pi)]dx$

Put $x + \pi = t \implies dx = dt$

$$\therefore \quad I=\int_{-\pi/2}^{\pi/2}[t^3+\cos^2(t+2\pi)]dt=\int_{-\pi/2}^{\pi/2}[t^3+\cos^2 t]dt$$

$$= 2\int_0^{\pi/2}\cos^2 t\ dt \qquad \left(\because \int_{-\pi/2}^{\pi/2} t^3dt = 0\right)$$

$$=\int_0^{\pi/2}(1+\cos 2t)dt = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{2}.$$

10. (b) : We are given that vectors lie in the same plane.

We know, vector

$\vec{L}=a\hat{i}+a\hat{j}+c\hat{k}$, $\vec{M}=\hat{i}+\hat{k}$, $\vec{N}=c\hat{i}+c\hat{j}+b\hat{k}$, are coplanar, if

$$[\vec{L}\,\vec{M}\,\vec{N}]=0\Rightarrow\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix}=0\Rightarrow c=\sqrt{ab}$$

$\therefore$ c is geometric mean of a and b.

11. (b) : Equation of the line through the points $(5, 1, a)$ and $(3, b, 1)$ is

$$\frac{x-5}{3-5}=\frac{y-1}{b-1}=\frac{z-a}{1-a}=\lambda$$

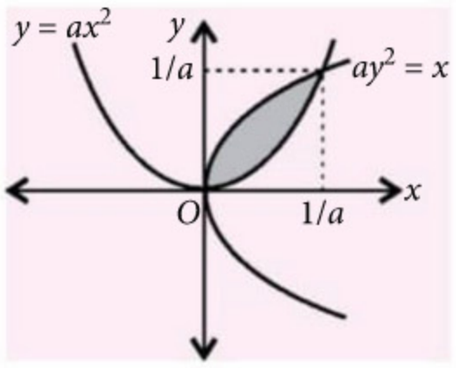
Now, it passes through $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$

$$\therefore \quad \frac{0-5}{-2}=\frac{(17/2-1)}{b-1}=\frac{(-13/2)-a}{1-a}=\lambda \implies \lambda=\frac{5}{2}$$

$\therefore \quad \dfrac{(17/2-1)}{b-1}=\dfrac{5}{2} \Rightarrow b=4$

and $\dfrac{(-13/2)-a}{1-a}=\dfrac{5}{2} \Rightarrow a=6$

12. (a) : The curves meet at (0, 0) and $\left(\dfrac{1}{a},\dfrac{1}{a}\right)$.



$\therefore \quad$ Required area $= \displaystyle\int_0^{1/a}\left(\sqrt{\frac{x}{a}}-ax^2\right)dx=1$

$\Rightarrow \quad \dfrac{2}{3a^2}-\dfrac{1}{3a^2}=1 \Rightarrow \dfrac{1}{3a^2}=1 \Rightarrow a=\dfrac{1}{\sqrt{3}}$

13. (d) : Let $I=\displaystyle\int\frac{(x\cos x+1)}{x\sqrt{2xe^{\sin x}+1}}dx$

Put $t^2 = 2xe^{\sin x} + 1 \Rightarrow t^2 - 1 = 2xe^{\sin x}$

Taking log on both sides, we get

$\log(t^2 - 1) = \log 2 + \log x + \sin x$

$\Rightarrow \quad \dfrac{2t}{t^2-1}dt=\dfrac{x\cos x+1}{x}dx$

$\therefore \quad I=\displaystyle\int\frac{2dt}{t^2-1}=\ln\left(\frac{t-1}{t+1}\right)+c$

$$=\ln\left(\frac{\sqrt{2xe^{\sin x}+1}-1}{\sqrt{2xe^{\sin x}+1}+1}\right)+c$$

14. (d) : The points are $A\left(0,\dfrac{8}{3}\right)$, $B(1, 3)$ and $C(82, 30)$

Slope of $AB = \dfrac{1}{3}$ = Slope of BC = Slope of AC

So, the given points lie on same line.

15. (b) : Given, centre lies on the line $y = 2$

$\therefore$ $C(\alpha, 2)$ and radius of circle = 5 units

$\therefore$ Equation of circle be

$(x - \alpha)^2 + (y - 2)^2 = 25$ ...(i)

Differentiating (i) w.r.t. to x, we get

$(x - \alpha) + (y - 2)y' = 0$

$\Rightarrow (x - \alpha)^2 = (y - 2)^2(y')^2$ ...(ii)

Putting (ii) in (i), we have

$(y - 2)^2(y')^2 + (y - 2)^2 = 25$

$\Rightarrow (y - 2)^2(y')^2 = 25 - (y - 2)^2$

16. (d) : Required probability

$$=\frac{({}^{10}C_0)^2+({}^{10}C_1)^2+({}^{10}C_2)^2+\ldots+({}^{10}C_{10})^2}{(2^{10})^2}=\frac{{}^{20}C_{10}}{2^{20}}$$

17. (c): Equation of chord is $y = mx + 1$

Equation of circle is $x^2 + y^2 = 1$

Joint equation of the curve through the intersection of line and circle be given by $x^2 + y^2 = (y - mx)^2$,

(By homonizing the equation)

$\Rightarrow \quad x^2(1 - m^2) + 2mxy = 0$

Now, $\tan\theta=\pm\dfrac{2\sqrt{h^2-ab}}{a+b}$, where $a=1-m^2$, $h=m, b=0$

$\therefore \quad \tan 45^\circ=\pm\dfrac{2\sqrt{m^2-0}}{1-m^2}$

$\Rightarrow 1(1 - m^2) = \pm 2m \Rightarrow m^2 \pm 2m - 1 = 0$

$\Rightarrow m = \pm 1 \pm \sqrt{2} \Rightarrow m = 1 \pm \sqrt{2}$ and $-1 \pm \sqrt{2}$

18. (a) : $f(x) = \sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

$3\sin^{-1}x \in \left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$

$\Rightarrow \quad \sin^{-1}x \in \left[-\dfrac{\pi}{6},\dfrac{\pi}{6}\right] \Rightarrow x \in \left[-\sin\dfrac{\pi}{6},\sin\dfrac{\pi}{6}\right]$

or $\quad x \in \left[-\dfrac{1}{2},\dfrac{1}{2}\right]$ which is obvious.

So, f is onto.

Now, say $x_1, x_2 \in \left[-\dfrac{1}{2},\dfrac{1}{2}\right]$ and $f(x_1) = f(x_2)$

$\Rightarrow 3x_1 - 4x_1^3 = 3x_2 - 4x_2^3$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

So, f is one-one also.

19. (a) : Consider the two curves $y^2 = 4ax$ and $x^2 = 4ay$.

Each is mirror image of the other in the line $x - y = 0$.

$\therefore$ The slope of the required line is -1.

$\therefore$ The normal $y = mx - 2am - am^3$ becomes

$x + y = 3a = \dfrac{3}{4}\left[\because a=\dfrac{1}{4}\right]$

The distance of the origin from it is $\dfrac{3}{4\sqrt{2}}$.

20. (d) : Let l, m, n be the D.C.'s of $\vec{r}$.

Then $l = m = n$ (given)

$\therefore \quad l^2+m^2+n^2=1 \Rightarrow 3l^2=1 \Rightarrow l=\dfrac{1}{\sqrt{3}}=m=n$

$\therefore \quad \vec{r}=|\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k})=6\left(\dfrac{1}{\sqrt{3}}\hat{i}+\dfrac{1}{\sqrt{3}}\hat{j}+\dfrac{1}{\sqrt{3}}\hat{k}\right)$

$=2\sqrt{3}(\hat{i}+\hat{j}+\hat{k})$.

21. (0) : $\frac{a_2}{a_1}=\frac{a_3}{a_2}=.....=\frac{a_n}{a_{n-1}}=r$

which means $a_n, a_{n+1}, a_{n+2} \in$ G.P. $\forall\ n \in N$

$\Rightarrow\ a_{n+1}^2 = a_n a_{n+2}$

$\Rightarrow\ 2\log a_{n+1} - \log a_n - \log a_{n+2} = 0$ ...(i)

Similarly, $2\log a_{n+4} - \log a_{n+3} - \log a_{n+5} = 0$...(ii)

and $2\log a_{n+7} - \log a_{n+6} - \log a_{n+8} = 0$ ...(iii)

Using $C_1 \to C_1 + C_3 - 2C_2$

we get $\Delta = 0$

22. (29) : Since order of vowels is not to change, the four vowels I, I, E, E are to be taken as similar.

Hence, the required no. is $\frac{6!}{4!}-1=29$

23. (1.92) : Since we are given rate per rupees, harmonic mean will give the correct answer.

$$\text{H.M.}=\frac{4}{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}=\frac{4\times12}{25}=1.92 \text{ kg per rupee}$$

24. (1) : We have, $f(x) = 2x^3 + bx^2 + cx$

Now, $f(1) = f(-1)$ and $f'\left(\frac{1}{2}\right)=0$

[$\because f(x)$ satisfies Rolle's theorem]

So, $f(1) = 2 + b + c$

$f(-1) = -2 + b - c$

$\therefore\ f(1) = f(-1) \Rightarrow c = -2$ ... (i)

Also, $f'(x) = 6x^2 + 2bx + c$

$$\Rightarrow\ f'\left(\frac{1}{2}\right)=\frac{3}{2}+b+c=0 \Rightarrow\ \frac{3}{2}+b-2=0$$

$$\Rightarrow\ b=\frac{1}{2}$$ ... (ii)

Hence, $|2b + c| = \left|\left(2\times\frac{1}{2}\right)+(-2)\right|=|-1|=1$

(using (i) and (ii))

25. (33) : Let $h = Ar + B$

$$\Rightarrow\ \frac{dh}{dt}=\frac{Adr}{dt}=\frac{3dr}{dt}\Rightarrow A=3$$ ...(i)

Given, $6 = A + B \Rightarrow B = 3$ [from (i)]

Hence, $h = 3r + 3$

Volume $(V) = \pi r^2 h = \pi r^2(3r + 3)$

$$\frac{dV}{dt}=3\pi[3r^2+2r]\frac{dr}{dt}$$

$$\Rightarrow\ n=3\pi(27+6)\frac{1}{3\pi}=33.$$

JEE 2020 ADVANCED

Exam on 27th September

PRACTICE PAPER

PAPER-1

SECTION-1

ONE OR MORE THAN ONE OPTION CORRECT TYPE

1. Let $f: [-1, 1]$ onto $[3, 5]$ be a linear polynomial. Which of the following can be true?

(a) $f\left(\frac{-1}{2}\right)=\frac{7}{2}$ (b) $f^{-1}\left(\frac{15}{4}\right)=\frac{1}{4}$

(c) $f(0) \neq 4$ (d) $f\left(\frac{1}{2}\right)+f\left(\frac{-1}{2}\right)=8$

2. A normal coin is tossed four times. Two events E and F are defined as

E : no two consecutive heads occur in 4 tosses.

F : At least 2 consecutive heads occur in 4 tosses.

The events E and F are

(a) equally likely

(b) mutually exclusive

(c) exhaustive

(d) such that one is twice as likely to occur as other.

3. If the expression $(1 + r\,i)^3$ is of the form of $s(1 + i)$ for some real 's' where 'r' is also real and $i=\sqrt{-1}$, then the value of 'r' can be

(a) $\cot\frac{\pi}{8}$ (b) $\sec\pi$

(c) $\tan\frac{\pi}{12}$ (d) $\tan\frac{5\pi}{12}$

4. Given $f(x) = \sum_{r=1}^{n}(x^r + x^{-r})^2;\ x \neq \pm 1$ and

$$g(x)=\begin{cases}\lim\limits_{n\to\infty}\left((f(x)-2n)x^{-2n-2}(1-x^2)\right); & x\neq\pm1\\ -1, & ;\ x=\pm1\end{cases}$$

then $g(x)$

(a) is discontinuous at $x = -1$

(b) is continuous at $x = 2$

(c) has a removable discontinuity at $x = 1$

(d) has an irremovable discontinuity at $x = 1$.

5. The expression $(\alpha\tan\gamma + \beta\cot\gamma)(\alpha\cot\gamma + \beta\tan\gamma) - 4\alpha\beta\cot^2 2\gamma$ is

(a) independent of α, β

(b) independent of γ

(c) dependent on γ

(d) dependent on α, β

6. The parabola $x = y^2 + ay + b$ intersect the parabola $x^2 = y$ at $(1, 1)$ at right angle. Which of the following is/are correct?

(a) $a = 4$ and $b = -4$

(b) $a = 2$ and $b = -2$

(c) Equation of the directrix for the parabola $x = y^2 + ay + b$ is $4x + 1 = 0$.

(d) Area enclosed by the parabola $x = y^2 + ay + b$ and its latus rectum is 1/24.

7. The position vectors of the vertices A, B and C of a tetrahedron are $(1, 1, 1)$, $(1, 0, 0)$ and $(3, 0, 0)$ respectively. The altitude from the vertex D to the opposite face ABC meets the median line through A of the ΔABC at a point E. If the length of side AD is 4 and volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, then the correct statement(s) is/are

(a) The altitude from the vertex D is 2.

(b) There is exactly one position for the point E.

(c) There can be two positions for the point E.

(d) Vector $\hat{j}-\hat{k}$ is normal to the plane ABC.

SECTION-2

INTEGER ANSWER TYPE

8. Let α, β, γ be distinct real numbers such that

$a\alpha^2 + b\alpha + c = (\sin\theta)\,\alpha^2 + (\cos\theta)\alpha$

$a\beta^2 + b\beta + c = (\sin\theta)\,\beta^2 + (\cos\theta)\beta$

$a\gamma^2 + b\gamma + c = (\sin\theta)\,\gamma^2 + (\cos\theta)\gamma$

(where $a, b, c, \in R$.)

Then the maximum value of the expression

$$\frac{a^2+b^2}{a^2+3ab+5b^2} \text{ is}$$

9. The integer satisfying the inequation

$$\sqrt{\log_{1/2}^2 x + 4\log_2\sqrt{x}} < \sqrt{2}(4-\log_{16} x^4) \text{ is}$$

10. In a sequence of circles C_1, C_2, C_3, ... C_n; the centres lie along positive x-axis with abscissae forming an arithmetic sequence of first term unity and common difference 3. The radius of these circles are in geometric sequence with first term unity and common ratio 2. If the tangent lines with slope m_1 and m_2 of C_3 are intersected at the centre of C_5, then compute the value of $5\,|m_1\, m_2|$.

11. For a positive constant t, let α, β be the roots of the quadratic equation $x^2 + t^2x - 2t = 0$. If the minimum value of $\int_{-1}^{2}\left(\left(x+\frac{1}{\alpha^2}\right)\left(x+\frac{1}{\beta^2}\right)+\frac{1}{\alpha\beta}\right)dx$ is $\sqrt{\frac{a}{b}}+c$ where $a, b, c \in N$, then find the least value of $(a+b+c)/10$.

12. Let $A=\begin{bmatrix}1 & \frac{3}{2}\\ 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}4 & -3\\ -2 & 2\end{bmatrix}$

and $C_r=\begin{bmatrix}r\cdot 3^r & 2^r\\ 0 & (r-1)3^r\end{bmatrix}$

be given matrices. If $\sum_{r=1}^{50} tr.((AB)^r C_r) = 3 + a\cdot 3^b$ where $tr.(A)$ denotes trace of matrix A, then find the value of $\frac{1}{20}(a+b)$. [Where a and b are relatively prime]

SECTION-3

MATRIX MATCH TYPE

Answer Q.13, Q.14, Q.15 by information given below:

Column I contains information about the family of curves

Column II contains the differential equation of the family of curve.

Column III contains relation between the order (O) and degree (D) of differential equation of the family of curve. where c_1, c_2, c_3, c_4, c_5, c_6 are arbitrary constant.

Column I		Column II		Column III	
(A)	$y = c_1\sin^2x + c_2\cos^2x + c_3\sin 2x + c_4\cos 2x$	(I)	$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$	(i)	$O^D + D^O = 3$
(B)	$y = (c_1 + c_2)\sin(x + c_3) - c_4e^{x+c_5+c_6}$	(II)	$\frac{1}{(1+x^2)}+\frac{1}{(1+y^2)}\frac{dy}{dx}=0$	(ii)	$O^D + D^O = 4$
(C)	$\sqrt{1+x^2}+\sqrt{1+y^2}=c_1(x\sqrt{1+y^2}+y\sqrt{1+x^2})$	(III)	$\frac{d^3y}{dx^3}+4\frac{dy}{dx}=0$	(iii)	$O^D + D^O = 2$
(D)	$y = c_1e^{3x} + c_2e^{5x}$	(IV)	$\frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$	(iv)	$2O^D + 3D^O = 9$

13. Which of the following options is the only correct combination?

(a) (A) - (III) - (ii) (b) (B) - (III) - (i)

(c) (C) - (I) - (iv) (d) (D) - (II) - (i)

14. Which of the following options is the only correct combination?

(a) (A) - (II) - (ii) (b) (B) - (III) - (iv)

(c) (C) - (III) - (iv) (d) (D) - (IV) - (i)

15. Which of the following options is the only incorrect combination?

(a) (B) - (I) - (iv) (b) (C) - (II) - (iii)

(c) (D) - (I) - (i) (d) (A) - (III) - (ii)

Answer Q.16, Q.17, Q.18 by information given below:

Column I contains information about the incident ray from a point on a curve at given point of the curve.

Column II contains the equation of reflected ray after the first reflection.

Column III contains the equation of reflected ray after the second reflection.

Column I		Column II		Column III	
(A)	A ray emanating from $(0, -\sqrt{5})$ incident on the curve $9x^2 + 4y^2 = 36$ at the point with abscissa 2.	(I)	$2x + y\sqrt{5} + 2\sqrt{5} = 0$	(i)	$x + 2y + \sqrt{5} = 0$
(B)	A ray emanating from $(\sqrt{5}, 0)$ incident on the curve $4x^2 + 9y^2 = 36$ at the point with ordinate -2	(II)	$2x + y - 2\sqrt{5} = 0$	(ii)	$4\sqrt{5}x + y + \sqrt{5} = 0$
(C)	A ray emanating from $(-\sqrt{5}, 0)$ incident on the curve $4x^2 + 5y^2 = 100$ at the point with ordinate $2\sqrt{5}$	(III)	$\sqrt{5}x + 2y - 2\sqrt{5} = 0$	(iii)	$x + 4\sqrt{5}y - \sqrt{5} = 0$

16. Which of the following options is the only correct combination?

(a) (A) - (I) - (ii) (b) (A) - (III) - (ii)

(c) (A) - (III) - (i) (d) (A) - (II) - (ii)

17. Which of the following options is the only correct combination?

(a) (B) - (II) - (i) (b) (B) - (I) - (iii)

(c) (B) - (III) - (i) (d) (B) - (II) - (ii)

18. Which of the following options is the correct combination?

(a) (C) - (II) - (iii) (b) (C) - (I) - (ii)

(c) (C) - (III) - (iii) (d) (C) - (II) - (i)

PAPER-2

SECTION-1

SINGLE OPTION CORRECT TYPE

1. Number of 4 digit numbers of the form $N = abcd$ which satisfy following three conditions

(i) $4000 \le N < 6000$

(ii) N is a multiple of 5

(iii) $3 \le b < c \le 6$ is equal to

(a) 12 (b) 18 (c) 24 (d) 48

2. The radii of the escribed circles of ΔABC are r_a, r_b and r_c respectively.

If $r_a + r_b = 3R$ and $r_b + r_c = 2R$, then the smallest angle of the triangle is

(a) $\tan^{-1}(\sqrt{2} - 1)$ (b) $\frac{1}{2}\tan^{-1}(\sqrt{3})$

(c) $\frac{1}{2}\tan^{-1}(\sqrt{2} + 1)$ (d) $\tan^{-1}(2 - \sqrt{3})$

3. The angle between pair of tangents drawn to the curve $7x^2 - 12y^2 = 84$ from $M(1, 2)$ is

(a) $2\tan^{-1}\frac{1}{2}$ (b) $2\tan^{-1}2$

(c) $2\left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right)$

(d) $2\tan^{-1}3$

4. The position vector of a point in which a line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $\vec{r}.(3\hat{i} - 5\hat{j} + 2\hat{k}) = 6$, is

(a) $(1, -1, -1)$ (b) $(-1, -1, 2)$

(c) $(4, 2, 2)$ (d) $\left(\frac{4}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$

5. Let $\binom{n}{k}$ represents the combination of 'n' things taken 'k' at a time, then the value of the sum $\binom{99}{97} + \binom{98}{96} + \binom{97}{95} + ... + \binom{3}{1} + \binom{2}{0}$ equals

(a) $\binom{99}{97}$ (b) $\binom{100}{98}$ (c) $\binom{99}{98}$ (d) $\binom{100}{97}$

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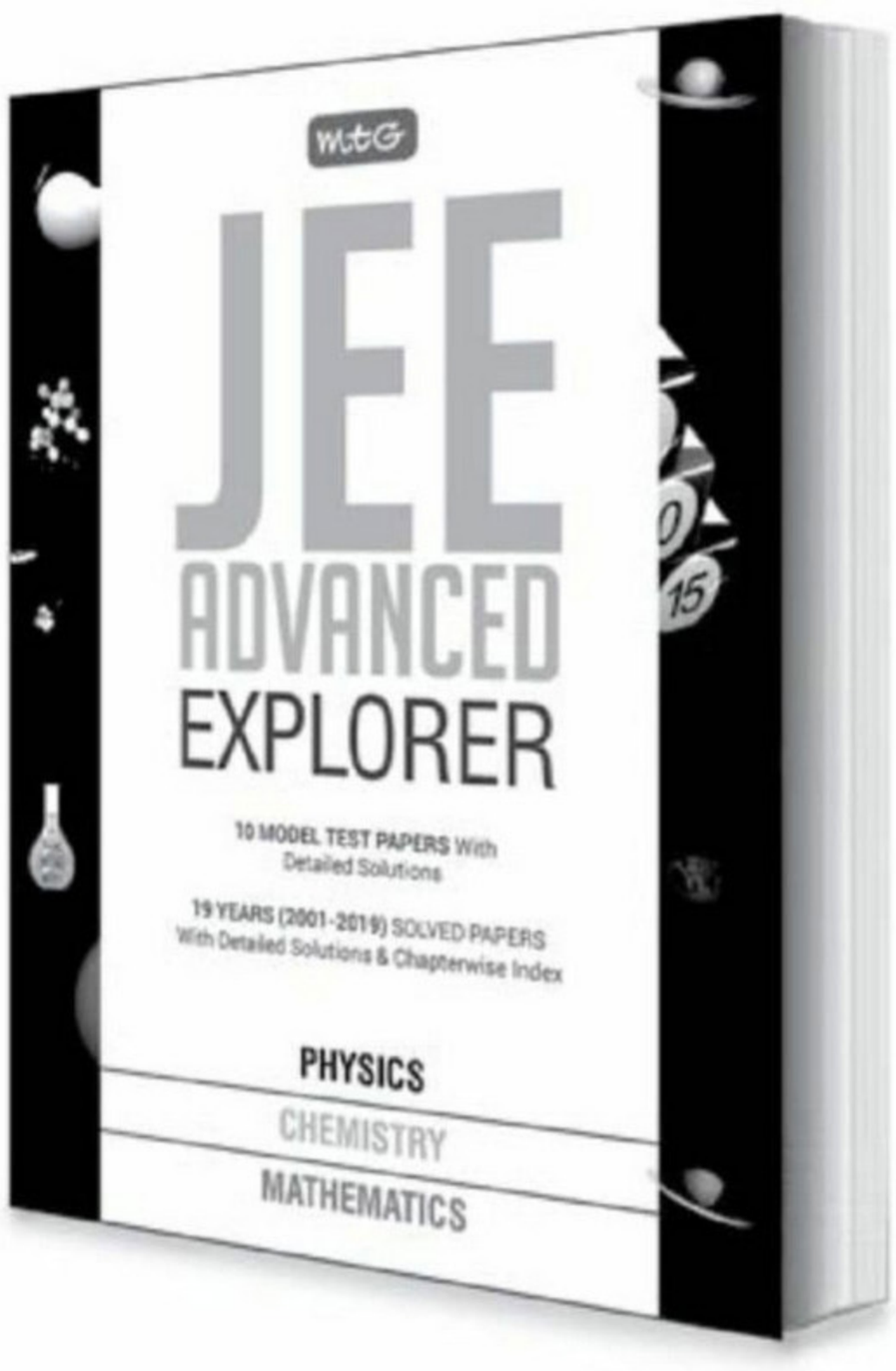

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6. Let $\alpha, \beta \in R$. If α, β^2 be the roots of quadratic equation $x^2 - px + 1 = 0$ and α^2, β be the roots of quadratic equation $x^2 - qx + 8 = 0$, then the value of $'r'$ if $\frac{r}{8}$ be arithmetic mean of p and q, is

(a) $\frac{83}{8}$ (b) $\frac{83}{4}$ (c) $\frac{83}{2}$ (d) 83

7. Let a, b, c, d are non-zero real numbers such that $6a + 4b + 3c + 3d = 0$, then the equation $ax^3 + bx^2 + cx + d = 0$ has

(a) atleast one root in $[-2, 0]$
(b) atleast one root in $[0, 2]$
(c) atleast two roots in $[-2, 2]$
(d) no root in $[-2, 2]$

SECTION-2

ONE OR MORE THAN ONE OPTION CORRECT TYPE

8. Consider the function $f(x) = \sin^5 x + \cos^5 x - 1$, $x \in \left[0, \frac{\pi}{2}\right]$. Which of the following is/are correct?

(a) f is monotonic increasing in $\left(0, \frac{\pi}{4}\right)$.

(b) f is monotonic decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(c) $\exists$ some $c \in \left(0, \frac{\pi}{2}\right)$ for which $f'(c) = 0$.

(d) The equation $f(x) = 0$ has two roots in $\left[0, \frac{\pi}{2}\right]$.

9. Which of the following statement(s) is/are correct?

(a) Rolle's theorem is applicable to the function $F(x) = 1 - \sqrt[5]{x^6}$ on the interval $[-1, 1]$.

(b) The domain of definition of the function $F(x) = \frac{\log_4(5-[x-1]-[x]^2)}{x^2+x-2}$ is $(-3, -2) \cup (-2, 1) \cup (1, 2)$
(where $[x]$ denotes the largest integer less than or equal to x)

(c) The value of a for which the function $F(\theta) = a \sin\theta + \frac{1}{3}\sin 3\theta$ has an extremum at $\theta = \pi/3$ is -2.

(d) The value of $\sum_{k=1}^{2010} \frac{\{x+k\}}{2010}$ is $\{x\}$
(where $\{x\}$ denotes the fractional part of x).

10. Let $I_n = \int_0^{\sqrt{3}} \frac{dx}{1+x^n}$ $(n = 1, 2, 3 \ldots\ldots)$ and $\lim_{n\to\infty} = I_0$ (say), then which of the following statement(s) is/are correct?

(a) $I_1 > I_0$ (b) $I_2 < I_0$
(c) $I_0 + I_1 + I_2 > 3$ (d) $I_0 + I_1 > 2$

11. A circle having its centre at $(2, 3)$ is cut orthogonally by the parabola $y^2 = 4x$. The possible intersection point(s) of these curves, can be

(a) $(3, 2\sqrt{3})$ (b) $(2, 2\sqrt{2})$
(c) $(1, 2)$ (d) $(4, 4)$

12. If $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 + (\sin^{-1}z)^2 = \frac{3\pi^2}{4}$, then the value of $(x - y + z)$ can be

(a) 1 (b) -1 (c) 3 (d) -3

13. Which of the following is/are true?

The circles $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 + 6x + 6y + 9 = 0$ are such that
(a) they do not intersect.
(b) they touch each other.
(c) their exterior common tangents are parallel.
(d) their interior common tangents are perpendicular.

14. A line L passing through the point $P(1, 4, 3)$, is perpendicular to both the lines

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{4} \text{ and } \frac{x+2}{3} = \frac{y-4}{2} = \frac{z+1}{-2}$$

If the position vector of point Q on L is (a_1, a_2, a_3) such that $(PQ)^2 = 357$, then $(a_1 + a_2 + a_3)$ can be

(a) 16 (b) 15 (c) 2 (d) 1

SECTION-3

COMPREHENSION TYPE

Paragraph for Question No. 15 and 16

Let $f(x) = 4x^4 - 24x^3 + 31x^2 + 6x - 8$ be a polynomial function. Suppose $f(x) = 0$ has four roots $\alpha < \beta < \gamma < \delta$. If sum of two of its roots vanishes, then

15. $\int \left(\frac{x-\delta}{x-\gamma}\right)^{\alpha+\beta+\delta} dx$ is

(a) $x - 8\ln|x-2| - \frac{24}{x-2} + \frac{16}{(x-2)^2} - \frac{16}{3(x-2)^3} + C$

(b) $x - 16\ln|x-2| - \frac{24}{(x-2)} + \frac{16}{(x-2)^2} - \frac{16}{(x-2)^3} + C$

(c) $x + 16\ln|x-2| - \frac{24}{x-2} + \frac{16}{(x-2)^2} - \frac{16}{3(x-2)^3} + C$

(d) $x + 8\ln|x-2| - \frac{24}{x-2} - \frac{16}{(x-2)^2} + \frac{16}{3(x-2)^3} + C$

16. $\int_{2\alpha}^{2\beta} \frac{x^{\delta+1} - 5x^{\gamma+1} + 2\beta|x| + 1}{x^2 + 4\beta|x| + 1} dx$ is

(a) ln2 (b) 2ln 2

(c) $\frac{1}{2}\ln 2$ (d) None of these

Paragraph for Question No. 17 and 18

Let a, b, c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a-\log b=\log 2$. Also the $a(1-x^2)+2bx+c(1+x^2)=0$ has two equal roots.

17. Measure of angle C is

(a) 30° (b) 45° (c) 60° (d) 90°

18. The value of $(\sin A+\sin B+\sin C)$ is equal to

(a) $\frac{5}{2}$ (b) $\frac{12}{5}$ (c) $\frac{8}{3}$ (d) 2

SOLUTIONS

PAPER-1

1. (a, b, d) : Let $f(x)=ax+b$

Case-I: f is increasing

$f(-1)=3$ and $f(1)=5$

$\Rightarrow -a+b=3$ and $a+b=5$

$\Rightarrow f(x)=x+4 \Rightarrow f^{-1}(x)=x-4$

Case-II: f is decreasing

$f(-1)=5$ and $f(1)=3$

$\Rightarrow -a+b=5$ and $a+b=3$

$\Rightarrow f(x)=4-x \Rightarrow f^{-1}(x)=4-x$

Now we can verify the options.

2. (a, b, c) : Sample space S is given by

$\{H\,T\,H\,H,\ H\,T\,T\,H,\ H\,T\,H\,T,\ H\,T\,T\,T,$
$T\,H\,H\,H,\ T\,H\,T\,H,\ T\,H\,H\,T,\ T\,H\,T\,T,$
$T\,T\,H\,H,\ T\,T\,T\,H,\ T\,T\,H\,T,\ T\,T\,T\,T,$
$H\,H\,H\,H,\ H\,H\,T\,H,\ H\,H\,H\,T,\ H\,H\,T\,T\}$

$\therefore \quad P(E)=\frac{8}{16}=\frac{1}{2}=P(F)$

3. (b, c, d) : $(1+r\,i)^3=s(1+i)$

$1+3r\,i+3r^2i^2+r^3i^3=s(1+i)$

$1-3r^2+i(3r-r^3)=s+si$

$\Rightarrow 1-3r^2=s=3r-r^3$

Hence $1-3r^2=3r-r^3$

$\Rightarrow r^3-3r^2-3r+1=0$

$\Rightarrow (r^3+1)-3r(r+1)=0$

$\Rightarrow (r+1)(r^2+1-r-3r)=0$

$\Rightarrow r=-1$ or $r^2-4r+1=0$

$\Rightarrow r=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm2\sqrt{3}}{2}$

$\Rightarrow r=2+\sqrt{3}$ or $2-\sqrt{3}$

4. (a, b, d) : $f(x)=\sum_{r=1}^{n}\left(x^{2r}+\frac{1}{x^{2r}}+2\right)$

$=\sum_{r=1}^{n}x^{2r}+\sum_{r=1}^{n}\frac{1}{x^{2r}}+2n$

$=(x^2+x^4+....+x^{2n})+\left(\frac{1}{x^2}+\frac{1}{x^4}+.....+\frac{1}{x^{2n}}\right)+2n$

$=\frac{x^2(1-x^{2n})}{(1-x^2)}+\frac{1}{x^2}\frac{\left(1-\frac{1}{x^{2n}}\right)}{1-\frac{1}{x^2}}+2n$

$f(x)=\frac{x^2(1-x^{2n})}{1-x^2}+\frac{(1-x^{2n})}{(1-x^2)x^{2n}}+2n$

$f(x)=\frac{(1-x^{2n})}{1-x^2}\left(x^2+\frac{1}{x^{2n}}\right)+2n,\ x\neq\pm1$

$\therefore\ (f(x)-2n)(1-x^2)=(1-x^{2n})\left(x^2+\frac{1}{x^{2n}}\right)$

Now consider $g(x)=\lim_{n\to\infty}\left((f(x)-2n)\,x^{-2n-2}(1-x^2)\right)$ for $x\neq\pm1$

$=\lim_{n\to\infty}(1-x^{2n})\left(x^2+\frac{1}{x^{2n}}\right)\cdot\frac{1}{x^{2n+2}};\ x\neq\pm1$

$=\lim_{n\to\infty}\frac{(1-x^{2n})(x^{2n+2}+1)}{(x^{2n})(x^{2n+2})}$

$=\lim_{n\to\infty}\left(-1+\frac{1}{x^{2n}}\right)\left(1+\frac{1}{x^{2n+2}}\right)$

Now, $g(x)=\begin{cases}-1 & ,|x|>1\\ \infty & ,|x|<1\\ -1 & ,x=\pm1\end{cases}$

Now clearly g has non removable infinite type of discontinuity at $x=1$ and -1

g is continuous at $x=2 \Rightarrow$ a, b, d

5. (b, d) : We have,

$(\alpha\tan\gamma+\beta\cot\gamma)(\alpha\cot\gamma+\beta\tan\gamma)-4\alpha\beta\cot^2 2\gamma$

$=\alpha^2+(\tan^2\gamma+\cot^2\gamma)\,\alpha\beta+\beta^2-4\alpha\beta\frac{\cos^2 2\gamma}{\sin^2 2\gamma}$

$=\alpha^2+\beta^2+\alpha\beta\left[\left(\frac{\sin^2\gamma}{\cos^2\gamma}+\frac{\cos^2\gamma}{\sin^2\gamma}\right)-\frac{4(\cos^2\gamma-\sin^2\gamma)^2}{4\sin^2\gamma\cos^2\gamma}\right]$

$=\alpha^2+\beta^2+\alpha\beta$

$\left[\frac{(\sin^4\gamma+\cos^4\gamma)-(\cos^4\gamma+\sin^4\gamma-2\sin^2\gamma\cos^2\gamma)}{\sin^2\gamma\cos^2\gamma}\right]$

$= \alpha^2 + \beta^2 + 2\alpha\beta = (\alpha + \beta)^2$, which is independent of γ and dependent on α, β.

6. (c, d) : Given curves, $x = y^2 + ay + b$; $x^2 = y$ passes through (1, 1)

$\therefore \quad 1 = 1 + a + b \Rightarrow a + b = 0$

Also, $1 = 2y\dfrac{dy}{dx} + a\dfrac{dy}{dx}$

$\left.\dfrac{dy}{dx}\right]_{(1,1)} = \dfrac{1}{2+a}$; For $y = x^2$; $\dfrac{dy}{dx} = 2x$; $\left.\dfrac{dy}{dx}\right]_{(1,1)} = 2$

$\because$ Given curves intersect at right angles

$\therefore \quad \dfrac{2}{2+a} = -1$

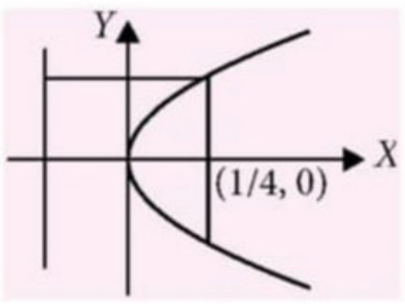
$\Rightarrow \quad -2 = 2 + a \quad \Rightarrow a = -4$ and $b = 4$

Hence parabola is

$y^2 - 4y + 4 = x$

$\Rightarrow \quad (y-2)^2 = x$

$\Rightarrow \quad Y^2 = X$ [Putting $y - 2 = Y$]



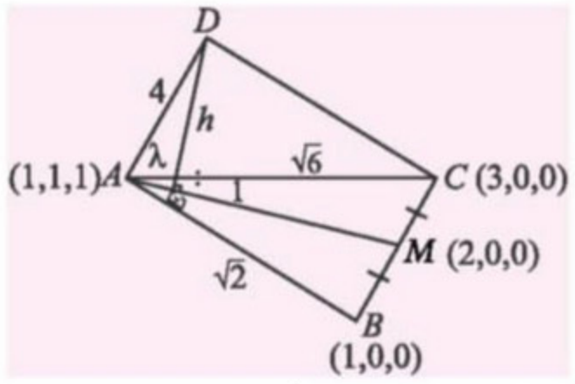
$\therefore \quad \text{Area} = \dfrac{1}{4}\cdot\dfrac{8}{3}\times\dfrac{1}{16} = \dfrac{1}{24}$

and directrix : $x = -\dfrac{1}{4} \Rightarrow 4x + 1 = 0$

7. (a, c, d) : Given $V = \dfrac{2\sqrt{2}}{3}$

Now $\dfrac{1}{3}\cdot\dfrac{1}{2}\begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{Vmatrix} h = \dfrac{2\sqrt{2}}{3}$

$\Rightarrow h\begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{Vmatrix} = 4\sqrt{2}$



[Note ABC is a right triangle $\rightarrow$ Area $= \dfrac{1}{2}(2)(\sqrt{2}) = \sqrt{2}$]

$h\left|\hat{i}(-1+1) + 2(\hat{j} - \hat{k})\right| = 4\sqrt{2}$

$\Rightarrow \quad h\left|\hat{j} - \hat{k}\right| = 2\sqrt{2} \Rightarrow \quad h = 2$

Let E divides AM in the ratio $\lambda : 1$

Hence $E : \left(\dfrac{2\lambda+1}{\lambda+1}, \dfrac{1}{\lambda+1}, \dfrac{1}{\lambda+1}\right)$

Now, $(AE)^2 + (DE)^2 = (AD)^2$

$\Rightarrow \quad \left(\dfrac{2\lambda+1}{\lambda+1} - 1\right)^2 + \left(1 - \dfrac{1}{\lambda+1}\right)^2 + \left(1 - \dfrac{1}{\lambda+1}\right)^2 + 4 = 16$

$\left(\dfrac{\lambda}{\lambda+1}\right)^2 + 2\left(\dfrac{\lambda}{\lambda+1}\right)^2 = 12 \Rightarrow \left(\dfrac{\lambda}{\lambda+1}\right)^2 = 4$

$\Rightarrow \quad \dfrac{\lambda}{\lambda+1} = 2 \text{ or } -2$

$\therefore \quad$ These are two positions for E which are (–1, 3, 3) and (3, –1, –1).

8. (2) : Let the equation be

$ax^2 + bx + c = (\sin\theta)\, x^2 + (\cos\theta)\, x$

$\Rightarrow \quad x^2(a - \sin\theta) + x(b - \cos\theta) + c = 0$

Then roots of equation are α, β, γ

Since quadratic has three roots then it must be an identity so.

$a - \sin\theta = 0$, $b - \cos\theta = 0$ and $c = 0$

$\therefore \quad a = \sin\theta$, $b = \cos\theta$, $c = 0$.

$\therefore \quad$ Required value of $\dfrac{a^2+b^2}{a^2+3ab+5b^2}$

$= \dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta + 3\sin\theta\cdot\cos\theta + 5\cos^2\theta}$

$\Rightarrow \dfrac{2}{6 + 4\cos 2\theta + 3\sin 2\theta}$

For maximum value, denominator should be minimum

$\therefore \quad 6 + 4\cos 2\theta + 3\sin 2\theta = 3 \pm 5$

$\Rightarrow \quad 6 - 5 = 1$

$\therefore \quad$ Maximum value of expression is 2.

9. (3) : $\sqrt{\log_{1/2}^2 x + 4\log_2\sqrt{x}} < \sqrt{2}\,(4 - \log_{16} x^4)$

$= \sqrt{\log_2^2 x + 2\log_2 x} < \sqrt{2}(4 - \log_2 x)$

Put $\log_2 x = t \quad \Rightarrow \sqrt{t^2 + 2t} < \sqrt{2}(4 - t)$ ...(i)

$\Rightarrow \quad t^2 + 2t \geq 0 \quad \Rightarrow \quad t(t+2) \geq 0$

$\Rightarrow \quad t \leq -2 \quad \text{or} \quad t \geq 0$ ...(ii)

$\Rightarrow \quad 4 - t \geq 0 \Rightarrow t \leq 4$ ...(iii)

Squaring (i), we get

$t^2 + 2t < 2(16 + t^2 - 8t)$

$\Rightarrow \quad 0 < t^2 - 18t + 32$

$\Rightarrow \quad (t - 16)(t - 2) > 0$

$\Rightarrow \quad t < 2 \quad \text{or} \quad t > 16$ ...(iv)

Using (ii), (iii) & (iv) we get $t \leq -2$ or $0 \leq t < 2$

$\Rightarrow \quad \log_2 x \leq -2 \quad \text{or} \quad 0 \leq \log_2 x < 2$

$\Rightarrow \quad x \leq \dfrac{1}{4} \quad \text{or} \quad 1 \leq x < 4$

$\Rightarrow \quad x \in \left(0, \dfrac{1}{4}\right] \cup [1, 4)$

The integral value of x is 3.

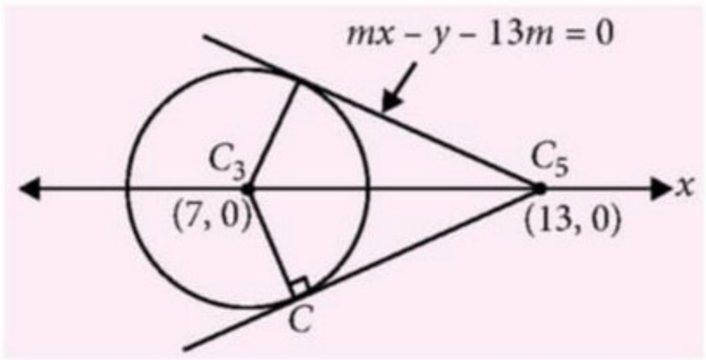
10. (4) : Centre: $C_n = 1 + (n-1)\,3$; $C_n = 3n - 2$

$C_5 = 13$; $C_3 = 7$; $C_5(13, 0)$; $C_3(7, 0)$

Radius: $R_n = ar^{n-1} = 2^{n-1} \quad \therefore R_3 = 4$

Line AB:

$y - 0 = m(x - 13)$; $mx - y - 13m = 0$



$$\Rightarrow \left|\frac{-6m}{\sqrt{m^2+1}}\right| = 4 \Rightarrow 9m^2 = 4m^2 + 4$$

$$\Rightarrow m^2 = \frac{4}{5} \Rightarrow m_1 = \frac{2}{\sqrt{5}} \text{ and } m_2 = \frac{-2}{\sqrt{5}}$$

Hence $5|m_1 m_2| = 5 \times \frac{4}{5} = 4$

11. (2) : If α and β are the roots of $x^2 + t^2x - 2t = 0$, then we have $\alpha + \beta = -t^2$ and $\alpha\beta = -2t$.

So $\frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \frac{t^2}{4} + \frac{1}{t}$

and $\frac{1}{(\alpha\beta)^2} + \frac{1}{\alpha\beta} = \frac{1}{4t^2} - \frac{1}{2t}$

Now $I = \int_{-1}^{2} \left(\left(x + \frac{1}{\alpha^2}\right)\left(x + \frac{1}{\beta^2}\right) + \frac{1}{\alpha\beta} \right) dx$

$$= \int_{-1}^{2} \left(x^2 + \left(\frac{t^2}{4} + \frac{1}{t}\right)x + \left(\frac{1}{4t^2} - \frac{1}{2t}\right) \right) dx = \frac{3t^2}{8} + \frac{3}{4t^2} + 3$$

Differentiating I w.r.t. 't', we get $\frac{dI}{dt} = \frac{3t}{4} - \frac{3}{2t^3} = 0$

So, we get $t = \pm \sqrt[4]{2}$, and since t is taken to be positive then

$$I_{\min} = I(\sqrt[4]{2}) = \frac{3\sqrt{2}}{4} + 3 = \sqrt{\frac{18}{16}} + 3 = \sqrt{\frac{9}{8}} + 3 = \sqrt{\frac{a}{b}} + c$$

$\therefore$ Least value of $(a + b + c)/10$ is 2.

12. (5) : $AB = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ $(AB)^1C_1 = C_1$, $(AB)^2C_2 = C_2$ and so on.

Also, $tr(C_r) = r \cdot 3^r + (r - 1) \cdot 3^r = (2r - 1) \cdot 3^r$

Now, $\sum_{r=1}^{50} tr((AB)^r C_r) = tr((AB)^1C_1) + tr((AB)^2C_2) + \ldots\ldots + tr((AB)^{50}C_{50}) = S$ (Let)

$\therefore$ $S = tr(C_1) + tr(C_2) + \ldots + tr(C_{50})$

$S = 1 \cdot 3^1 + 3 \cdot 3^2 + 5 \cdot 3^3 + \ldots\ldots + 99 \cdot 3^{50}$

$3S = 1 \cdot 3^2 + 3 \cdot 3^3 + \ldots\ldots + 97 \cdot 3^{50} + 99 \cdot 3^{51}$

$-2S = 1 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \ldots\ldots + 2 \cdot 3^{50} - 99 \cdot 3^{51}$

$= -3 + 2 \cdot 3 + 2 \cdot 3^2 + \ldots\ldots + 2 \cdot 3^{50} - 99 \cdot 3^{51}$

$= -3 + 2 \cdot \frac{3 \cdot (3^{50} - 1)}{3 - 1} - 99 \cdot 3^{51} = -3 + 3^{51} - 3 - 99 \cdot 3^{51}$

$= -6 - 98 \cdot 3^{51} \Rightarrow S = 3 + 49 \cdot 3^{51}$

$\therefore$ $a + b = 100$

Hence $\frac{1}{20}(a + b) = 5$

(13 – 15) : 13. (a) 14. (d) 15. (c)

The correct set of combination from the above table is :
(A) - (III) - (ii); (B) - (I) - (iv) ; (C) - (II) - (iii) ; (D) - (IV) - (i)

(A) $\because$ $y = c_1\left(\frac{1 - \cos 2x}{2}\right) + c_2\left(\frac{1 + \cos 2x}{2}\right)$

$+ c_3 \sin 2x + c_4 \cos 2x = A + B \sin 2x + C \cos 2x$

$\therefore$ $\frac{dy}{dx} = 2B \cos 2x - 2C \sin 2x$ ...(i)

$\Rightarrow$ $\frac{d^2y}{dx^2} = -4B \sin 2x - 4C \cos 2x$

$\Rightarrow$ $\frac{d^3y}{dx^3} = -8B \cos 2x + 8C \sin 2x = -4\frac{dy}{dx}$ [From Eq. (i)]

$\Rightarrow$ $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = 0$ $\therefore$ $O = 3$, $D = 1$

(B) $\because$ $y = (c_1 + c_2) \sin(x + c_3) - c_4 e^{c_5 + c_6} \cdot e^x$

or $y = A \sin(x + B) + Ce^x$ .....(i)

$\therefore$ $\frac{dy}{dx} = A\cos(x + B) + Ce^x$ .....(ii)

Subtracting Eq. (i) from Eq. (ii), we get

$\frac{dy}{dx} - y = A \cos(x + B) - A \sin(x + B)$ ....(iii)

$\Rightarrow$ $\frac{d^2y}{dx^2} - \frac{dy}{dx} = -A \sin(x + B) - A \cos(x + B)$

$\Rightarrow$ $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = -A \cos(x + B) + A \sin(x + B)$

$= -\left(\frac{dy}{dx} - y\right)$ [from Eq. (iii)]

$\Rightarrow$ $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$ $\therefore$ $O = 3$, $D = 1$

(C) Put $x = \tan\theta$, $y = \tan\phi$

Then, $(\sec\theta + \sec\phi) = c_1(\tan\theta \sec\phi + \tan\phi \sec\theta)$

$\Rightarrow$ $\left(\frac{\cos\theta + \cos\phi}{\cos\theta\cos\phi}\right) = c_1\left(\frac{\sin\theta + \sin\phi}{\cos\theta\cos\phi}\right)$

$\Rightarrow$ $2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$

$= c_1.2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$

$\Rightarrow \cot\left(\frac{\theta+\phi}{2}\right)=c_1 \Rightarrow \frac{\theta+\phi}{2}=\cot^{-1}c_1$

or $\frac{1}{(1+x^2)}+\frac{1}{(1+y^2)}\frac{dy}{dx}=0 \quad \therefore O=1,\ D=1$

(D) $y = c_1e^{3x} + c_2e^{5x}$ or $c_1e^{3x} + c_2e^{5x} - y = 0$ ...(i)

$\therefore \quad \frac{dy}{dx} = 3c_1e^{3x} + 5c_2e^{5x}$

or $3c_1e^{3x} + 5c_2e^{5x} - \frac{dy}{dx} = 0$ ...(ii)

Again differentiating both sides w.r.t. x,

then $9c_1e^{3x} + 25c_2e^{5x} - \frac{d^2y}{dx^2} = 0$ ...(iii)

From Eqs. (i), (ii) and (iii) by eliminating c_1, c_2, c_3, we get

$$\begin{vmatrix} 1 & 1 & y \\ 3 & 5 & \frac{dy}{dx} \\ 9 & 25 & \frac{d^2y}{dx^2} \end{vmatrix} = 0$$

Expanding w.r.t. R_1, then

$$\left(5\frac{d^2y}{dx^2}-25\frac{dy}{dx}\right)-1\left(3\frac{d^2y}{dx^2}-9\frac{dy}{dx}\right)+y(75-45)=0$$

$\Rightarrow 2\frac{d^2y}{dx^2}-16\frac{dy}{dx}+30y=0 \ \therefore \frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$

$\therefore \quad O = 2,\ D = 1$

(16 - 18) : 16. (b) 17. (b) 18. (d)

The correct set of combination from the above table is :

(A) - (III) - (ii); (B) - (I) - (iii); (C) - (II) - (i)

(A) Given $9x^2 + 4y^2 = 36$ or $\frac{x^2}{4}+\frac{y^2}{9}=1$

i.e., ellipse along y-axis. If e be the eccentricity, then

$4 = 9(1 - e^2)$

$\Rightarrow \quad e=\frac{\sqrt{5}}{3}$

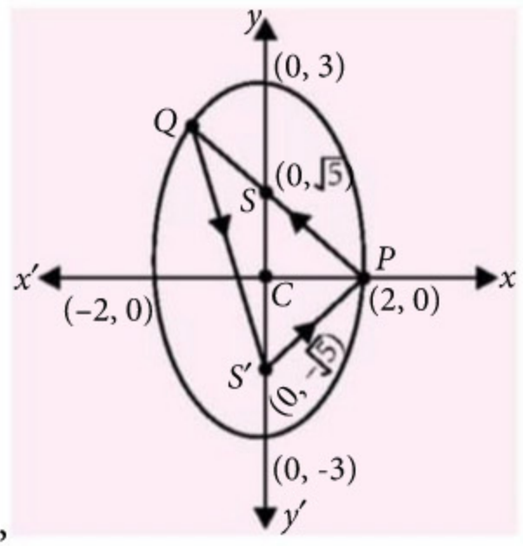
Then, foci $= (0, \pm \sqrt{5})$

and $P \equiv (2, 0)$

$\therefore$ Equation of first reflected ray PQ is

$x\sqrt{5}+2y-2\sqrt{5}=0$ ...(i)

Solving (i) and $9x^2 + 4y^2 = 36$,



we get $Q=\left(-\frac{4}{7}, \frac{9\sqrt{5}}{7}\right)$

$\therefore$ Equation of second reflected ray is

$y+\sqrt{5}=\frac{\frac{9\sqrt{5}}{7}+\sqrt{5}}{-\frac{4}{7}-0}(x-0)$ (pass through Q, S')

or $4\sqrt{5}x + y + \sqrt{5} = 0$

(B) $4x^2 + 9y^2 = 36$

or $\frac{x^2}{9}+\frac{y^2}{4}=1$

i.e., ellipse along x-axis.

If e be the eccentricity, then

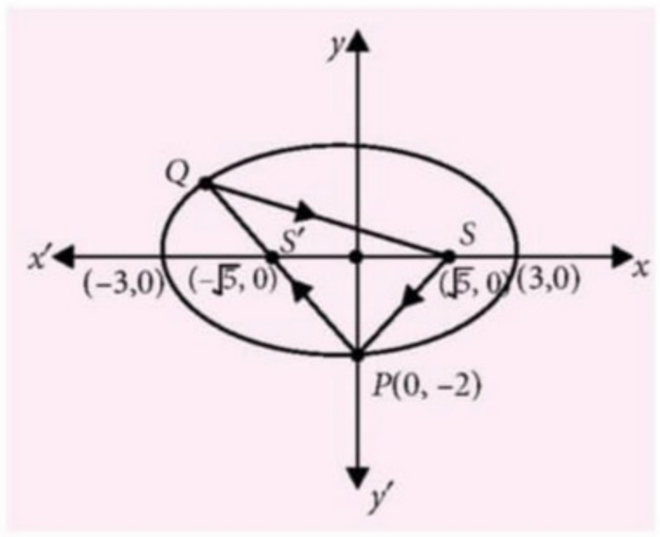
$4 = 9(1 - e^2)$

or $e=\frac{\sqrt{5}}{3}$

Then,

foci $= (\pm \sqrt{5}, 0)$

and $P = (0, -2)$



$\therefore$ Equation of first reflected ray PQ is $\frac{x}{-\sqrt{5}}+\frac{y}{-2}=1$

or $2x + y\sqrt{5} + 2\sqrt{5} = 0$ ...(ii)

Solving (ii) and $4x^2 + 9y^2 = 36$,

we get $Q \equiv \left(-\frac{9\sqrt{5}}{7}, \frac{4}{7}\right)$

$\therefore$ Equation of the second reflected ray is

$y-0=\frac{\frac{4}{7}-0}{\frac{-9\sqrt{5}}{7}-\sqrt{5}}(x-\sqrt{5})$ (Pass through Q, S)

$\Rightarrow \quad y=-\frac{1}{4\sqrt{5}}(x-\sqrt{5})$ or $x+4\sqrt{5}y-\sqrt{5}=0$

(C) $\because$ Given $4x^2 + 5y^2 = 100$ or $\frac{x^2}{25}+\frac{y^2}{20}=1$

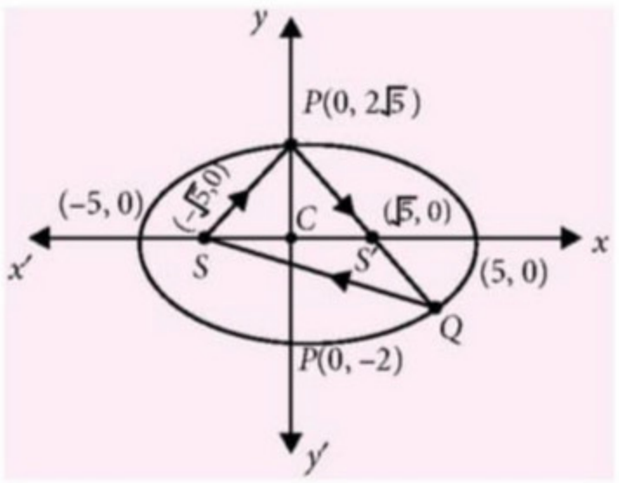
i.e., ellipse along x-axis.

If e be the eccentricity then

$20 = 25\ (1 - e^2)$

$\Rightarrow \quad \frac{4}{5}=1-e^2$

$\therefore \quad e=\frac{1}{\sqrt{5}}$



Then, foci $(\pm\sqrt{5}, 0)$ and $P \equiv (0, 2\sqrt{5})$

$\therefore$ Equation of first reflected ray PQ is

$$\frac{x}{\sqrt{5}}+\frac{y}{2\sqrt{5}}=1 \quad \text{or } 2x + y - 2\sqrt{5} = 0 \qquad \ldots(iii)$$

Now, solving (iii) and $4x^2 + 5y^2 = 100$

we get $Q \equiv \left(\frac{5\sqrt{5}}{3}, -\frac{4\sqrt{5}}{3}\right)$

$\therefore$ Equation of the second reflected ray is

$$y-0=\frac{-4\sqrt{5}/3-0}{5\sqrt{5}/3+\sqrt{5}}(x+\sqrt{5}) \text{ (pass through } Q,\ S)$$

$$\Rightarrow \quad y=-\frac{1}{2}(x+\sqrt{5})$$

or $x+2y+\sqrt{5}=0 \quad \therefore \quad S: x+2y+\sqrt{5}=0$

PAPER-2

1. (c) : We have $N = \boxed{a}\boxed{b}\boxed{c}\boxed{d}$

First place a can be filled in 2 ways *i.e.* 4, 5 $(4000 \le N < 6000)$

For b and c, total possibilities are '6' $(3 \le b < c \le 6)$ *i.e.* 34, 35, 36, 45, 46, 56

Last place d can be filled in 2 ways *i.e.* 0, 5 (N is a multiple of 5)

Hence total numbers = $2 \times 6 \times 2 = 24$

2. (b) : We have $r_a + r_b = 3R$

$$\Rightarrow \quad \frac{\Delta}{s-a}+\frac{\Delta}{s-b}=3R=\frac{3abc}{4\Delta} \quad \left(\because R=\frac{abc}{4\Delta}\right)$$

$$\Rightarrow \quad \frac{\Delta(s-b+s-a)}{(s-a)(s-b)}=\frac{3abc}{4\Delta}$$

$$\Rightarrow \quad \frac{c\Delta}{(s-a)(s-b)}=\frac{3abc}{4\Delta} \quad \Rightarrow \quad \frac{\Delta^2}{(s-a)(s-b)}=\frac{3ab}{4}$$

$\Rightarrow \quad 4s(s - c) = 3ab \quad \Rightarrow (a + b + c)(a + b - c) = 3ab$

$\Rightarrow \quad (a + b)^2 - c^2 = 3ab \quad \Rightarrow \quad a^2 + b^2 - c^2 = ab$

$\Rightarrow \quad c^2 = a^2 + b^2 - ab$

$\Rightarrow \quad a^2 + b^2 - 2ab \cos C = a^2 + b^2 - ab$

$(\because \quad c^2 = a^2 + b^2 - 2ab \cos C)$

$$\Rightarrow \quad \cos C=\frac{1}{2} \Rightarrow \angle C = 60° \qquad \ldots(1)$$

Similarly from $r_b + r_c = 2R$

$$\Rightarrow \quad \frac{\Delta}{s-b}+\frac{\Delta}{s-c}=2R \quad \Rightarrow \quad \frac{\Delta(2s-b-c)}{(s-b)(s-c)}=\frac{2abc}{4\Delta}$$

$$\Rightarrow \quad \frac{2\Delta^2}{(s-b)(s-c)}=bc \quad \Rightarrow 2s(s - a) = bc$$

$\Rightarrow \quad (b + c + a)(b + c - a) = 2bc$

$\Rightarrow \quad (b + c)^2 - a^2 = 2bc \quad \Rightarrow \quad b^2 + c^2 = a^2$

$\Rightarrow \quad \angle A = 90° \quad \Rightarrow \quad \angle B = 30°$

3. (c) : The director circle of given hyperbola $\frac{x^2}{12}-\frac{y^2}{7}=1$, is $x^2 + y^2 = 5$ and given point $M(1, 2)$ lies on it.

$\Rightarrow$ The angle between pair of tangents $= \frac{\pi}{2}$.

As $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\tan^{-1}1=\frac{\pi}{4}$

$$\Rightarrow \quad 2\left(\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right)=\frac{\pi}{2}$$

4. (d) : Vector perpendicular to $2x - y - z = 4$ is $\vec{n}=2\hat{i}-\hat{j}-\hat{k}$

Also line is collinear with $\vec{n}$.

So, equation of line is

$$\vec{r}=\lambda(2\hat{i}-\hat{j}-\hat{k}) \qquad \ldots(1)$$

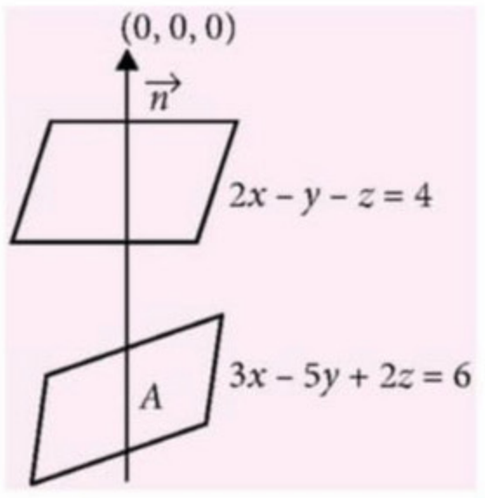
Now line (1) meet the plane

$\vec{r}\cdot(3\hat{i}-5\hat{j}+2\hat{k})=6$

So, $\lambda(6 + 5 - 2) = 6$

$$\Rightarrow \quad \lambda=\frac{2}{3}$$

Hence P.V. of the point 'A' is $\frac{2}{3}(2\hat{i}-\hat{j}-\hat{k})$



5. (d) : $S = {}^2C_0 + {}^3C_1 + {}^4C_2 + {}^5C_3 + \ldots\ldots + {}^{99}C_{97}$

$\Rightarrow S = \underbrace{{}^3C_0 + {}^3C_1} + {}^4C_2 + {}^5C_3 + \ldots\ldots + {}^{99}C_{97}$

$= \underbrace{{}^4C_1 + {}^4C_2} + {}^5C_3 + \ldots\ldots + {}^{99}C_{97} = {}^5C_2 + {}^5C_3 + \ldots\ldots$

$\therefore \quad S = {}^{100}C_{97} \Rightarrow S=\frac{100\cdot99\cdot98}{3\cdot2}=161700$

6. (d) : For the equation $x^2 - px + 1 = 0$, the product of roots, $\alpha\beta^2 = 1$

and for the equation $x^2 - qx + 8 = 0$, the product of roots $\alpha^2\beta = 8$

Hence, $(\alpha\beta^2)(\alpha^2\beta) = 8 \Rightarrow \alpha^3\beta^3 = 8 \Rightarrow \alpha\beta = 2$

$\therefore$ From $\alpha\beta^2 = 1$, we have $\beta = \frac{1}{2}$ and from $\alpha^2\cdot\beta = 8$, we have $\alpha = 4$

Now, $p = \alpha + \beta^2 = 4 + \frac{1}{4}=\frac{17}{4}$

and $q=\alpha^2+\beta=16+\frac{1}{2}=\frac{33}{2}$

$\because \quad \frac{r}{8}$ is arithmetic mean of p and q

$\therefore \frac{r}{8}=\frac{p+q}{2} \Rightarrow r=4(p+q)=4\left(\frac{17}{4}+\frac{33}{2}\right)=17+66=83$

7. (b) : We have $ax^3 + bx^2 + cx + d = 0$

Let $f(x)=\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+dx+e \quad \therefore \ f(0) = e$

$f(2) = 4a + \frac{8b}{3} + 2c + 2d + e$

$=\frac{(12a+8b+6c+6d)}{3}+e=\frac{2}{3}(6a+4b+3c+3d)+e=0$

$\Rightarrow f(2) = e$

$\therefore$ By Rolle's theorem, there exist atleast one value of $x \in (0, 2)$ such that $f'(x) = 0$

$\Rightarrow$ The equation $ax^3 + bx^2 + cx + d = 0$ has atleast one real root in $[0, 2]$

8. (c, d) : We have $f'(x) = 5 \sin^4 x \cos x - 5 \cos^4 x \sin x$
$= 5 \sin x \cos x(\sin x - \cos x)(1 + \sin x \cos x)$

$\therefore \quad f'(x) = 0$ at $x=\frac{\pi}{4}$. Also $f(0)=f\left(\frac{\pi}{2}\right)=0$

Hence $\exists$ some $c \in \left(0, \frac{\pi}{2}\right)$ for which $f'(c) = 0$ (By Rolle's Theorem)

Also in $\left(0, \frac{\pi}{4}\right)$ f is decreasing and in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ f is increasing

$\Rightarrow$ minimum at $x = \frac{\pi}{4}$

As $f(0) = f\left(\frac{\pi}{2}\right) = 0 \Rightarrow$ 2 roots

9. (a, d) : (a) We have, $F(x)=\left(1-x^{6/5}\right)$

Now, $F'(x)=\frac{-6}{5}x^{\frac{1}{5}}$ exist $\forall\, x \in (-1, 1)$

Also, $F(-1) = 0 = F(1)$

Hence Rolle's theorem is applicable to the function $F(x)$.

(b) For domain of $F(x)$,

$5 - [x] + 1 - [x]^2 > 0$ and $x^2 + x - 2 \neq 0$

$\Rightarrow (x + 2)(x - 1) \neq 0 \Rightarrow x \neq -2, 1$

Now $[x]^2 + [x] - 6 < 0 \Rightarrow ([x] + 3)([x] - 2) < 0$

$\Rightarrow -3 < [x] < 2 \Rightarrow -2 \le x < 2$

$\therefore$ Domain $= (-2, 1) \cup (1, 2)$

(c) We have, $F(\theta) = a \sin\theta + \frac{1}{3}\sin 3\theta$

As $F(\theta)$ has an extremum at $\theta = \frac{\pi}{3}$, so

$\Rightarrow a \cos\theta + \cos 3\theta = 0$ at $\theta = \frac{\pi}{3} \Rightarrow \frac{a}{2}-1=0$

$\Rightarrow \frac{a}{2}-1 \Rightarrow a=2$

(d) We have $\sum_{k=1}^{2010}\frac{\{x+k\}}{2010}=\frac{\{x+1\}}{2010}+\frac{\{x+2\}}{2010}+.....$

$+\frac{\{x+2010\}}{2010}=\frac{2010\{x\}}{2010}=\{x\}$

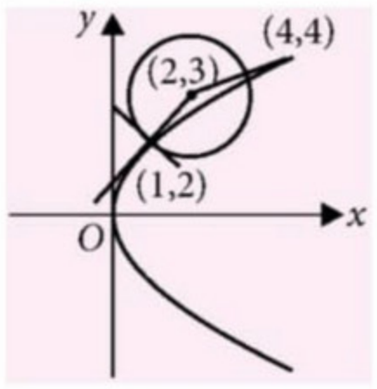
10. (a, c, d) : $I_1 = \ln(1+\sqrt{3})$

$I_2=\frac{\pi}{3}$

$I_0 = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\left[\int_0^1 \frac{dx}{1+x^n}+\underbrace{\int_1^{\sqrt{3}}\frac{dx}{1+x^n}}_{zero}\right]=\int_0^1 dx=1.$

Hence $I_0 = 1$. Now verify all alternatives.

11. (c, d) : Any tangent to the parabola $y^2 = 4x$, is
$yt = x + t^2$ at $(t^2, 2t)$
If it passes through the centre (2, 3) of the circle,
then $t^2 - 3t + 2 = 0 \Rightarrow t = 1, 2$
$\therefore$ The point can be (1, 2) or (4, 4)
(one circle will be of radius $\sqrt{2}$ and other will be of radius $\sqrt{5}$)



12. (a, b, c, d) :

$\because \frac{-\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \forall -1 \le x \le 1$

$\therefore 0 \le (\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 \le \frac{3\pi^2}{4}$

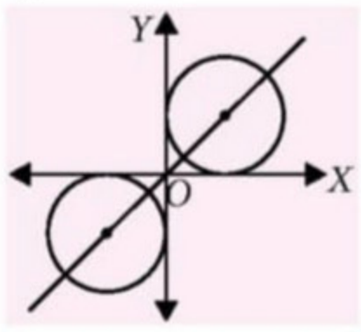
$\therefore (\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 = \frac{3\pi^2}{4}$ is possible if $x, y, z \in \{-1, 1\}$

$\therefore$ Possible values of $x - y + z$ from the ordered triplet (x, y, z) are as follows

(x, y, z)	$x - y + z$	(x, y, z)	$x - y + z$
$(-1, -1, -1)$	-1	$(1, 1, 1)$	1
$(-1, 1, 1)$	-1	$(1, -1, -1)$	1
$(1, -1, 1)$	3	$(-1, 1, -1)$	-3
$(1, 1, -1)$	-1	$(-1, -1, 1$	1

$\therefore$ Set of values of $x - y + z$ is $\{\pm 1, \pm 3)$

13. (a, c, d) : Centre (3, 3) with radius = 3 and centre (–3, –3) with radius = 3.



14. (b, d): Equation of the line passing through $P(1,\ 4,\ 3)$ and direction ratio's a, b & c is

$$\frac{x-1}{a}=\frac{y-4}{b}=\frac{z-3}{c} \qquad ...(i)$$

Since (i) is perpendicular to $\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{4}$

and $\frac{x+2}{3}=\frac{y-4}{2}=\frac{z+1}{-2}$

Hence $2a + b + 4c = 0$ and $3a + 2b - 2c = 0$

$$\therefore\ \frac{a}{-2-8}=\frac{b}{12+4}=\frac{c}{4-3} \Rightarrow \frac{a}{-10}=\frac{b}{16}=\frac{c}{1}$$

Hence the equation of the line is

$$\frac{x-1}{-10}=\frac{y-4}{16}=\frac{z-3}{1} \qquad ...(ii)$$

Now any point Q on (2) can be taken as $(1 - 10\lambda, 16\lambda + 4, \lambda + 3)$

$\therefore$ Distance of Q from $P(1, 4, 3) = (10\lambda)^2 + (16\lambda)^2 + \lambda^2 = 357$

$\Rightarrow\ (100 + 256 + 1)\lambda^2 = 357$

$\Rightarrow\ \lambda = 1$ or -1

$\therefore$ Q is $(-9, 20, 4)$ or $(11, -12, 2)$

Hence, $a_1 + a_2 + a_3 = 15$ or 1

(15 – 16) : 15. (a) 16. (b)

Sum of all four roots = 6; Sum of two roots = 0

$\Rightarrow$ sum of other two roots = 6

The factors corresponding to these roots are of the type $x^2 - 0x + p$ and $x^2 - 6x + q$ *i.e.*, $x^2 + p$ and $x^2 - 6x + q$

Thus, we have $4x^4 - 24x^3 + 31x^2 + 6x - 8 \equiv 4(x^2 + p)(x^2 - 6x + q)$

Equating like powers of x, we get

$$31 = 4(p + q) \Rightarrow p + q = \frac{31}{4}$$

Also, $6 = 4(-6p) \Rightarrow p = -\frac{1}{4}$ $\quad\therefore\ q = 8$

$\therefore$ Equation $f(x) = 0$ can be written as

$$\left(x^2-\frac{1}{4}\right)(x^2-6x+8)=0 \Rightarrow x=\pm\frac{1}{2} \text{ or } x=2,4$$

$$\therefore\ \alpha=-\frac{1}{2},\ \beta=\frac{1}{2},\ \gamma=2,\ \delta=4$$

(i) Now $\int\left(\frac{x-\delta}{x-\gamma}\right)^{\alpha+\beta+\delta}dx=\int\left(\frac{x-4}{x-2}\right)^{-\frac{1}{2}+\frac{1}{2}+4}dx$

$$=\int\left(\frac{x-4}{x-2}\right)^{4}dx$$

Let $x - 2 = t$, then

$$t\int\frac{(t-2)^4}{t^4}dt=\int\frac{t^4-8t^3+24t^2-32t+16}{t^4}dt$$

$$=\int\left(1-\frac{8}{t}+\frac{24}{t^2}-\frac{32}{t^3}+\frac{16}{t^4}\right)dt$$

$$=t-8\ln|t|-\frac{24}{t}+\frac{16}{t^2}-\frac{16}{3t^3}+C$$

$$=x-8\ln|x-2|-\frac{24}{x-2}+\frac{16}{(x-2)^2}-\frac{16}{3(x-2)^3}+C$$

(ii) $\int_{2\alpha}^{2\beta}\frac{x^{\delta+1}-5x^{\gamma+1}+2\beta|x|+1}{x^2+4\beta|x|+1}dx$

(putting α, β, γ and δ)

$$=\int_{-1}^{1}\frac{x^5-5x^3+|x|+1}{x^2+2|x|+1} \quad (\because\ x^5-5x^3 \text{ are odd function.})$$

$$=\int_{-1}^{1}\frac{|x|+1}{(|x|+1)^2}=2\int_{0}^{1}\frac{dx}{(|x|+1)}=2\int_{0}^{1}\frac{dx}{1+x}$$

$$= 2[\ln|(1 + x)|]_0^1 = 2\ln 2$$

(17 – 18) : 17. (d) 18. (b)

Given $\log\left(\frac{a+c}{a}\right)+\log\left(\frac{a}{b}\right)=\log 2$

$$\Rightarrow\ \log\left(\frac{a+c}{b}\right)=\log 2 \Rightarrow\ a + c = 2b \qquad ...(i)$$

Also, $a - ax^2 + 2bx + c + cx^2 = 0$ has equal roots

$\Rightarrow\ (c - a)x^2 + 2bx + (c + a) = 0$ has equal roots

$\therefore\ D = 0 \Rightarrow 4b^2 - 4(c^2 - a^2) = 0$

$\therefore\ b^2 = c^2 - a^2$ ...(ii)

$\Rightarrow$ Triangle is right angled at $C \Rightarrow\ \angle C = 90°$

From (i) and (ii), $b^2 = (c - a)(c + a) = (c - a)\ 2b$

$\Rightarrow\ 2(c - a) = b$ ...(iii)

As $\angle C = 90° \Rightarrow \angle A + \angle B = 90°$

From (iii) using Sine law,

$2(\sin C - \sin A) = \sin B$ ...(iv)

Now, $C = 90° \Rightarrow \sin C = 1$

Also, $A + B = 90° \Rightarrow B = 90° - A$

$\therefore$ (iv) becomes $2(1 - \sin A) = \sin(90 - A) = \cos A$

Squaring both sides, we get

$4(1 - \sin A)^2 = \cos^2 A = (1 - \sin^2 A)$

$\Rightarrow\ 4(1 - \sin A) = (1 + \sin A) \Rightarrow 3 = 5\sin A$

$\Rightarrow\ \sin A = 3/5$

Now, $B = 90 - A$

$\Rightarrow\ \sin B = \cos A = 4/5$ and $\sin C = 1$

$$\therefore\ \sin A+\sin B+\sin C=\frac{3}{5}+\frac{4}{5}+1=\frac{12}{5}$$

CONCEPT BOOSTERS

Class XI

Ellipse and Hyperbola

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

ELLIPSE

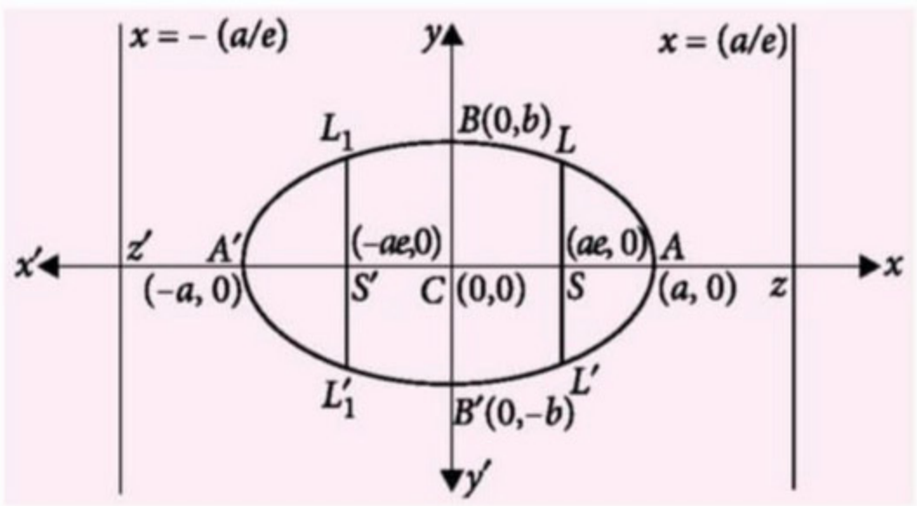
- The Ellipse is a conic whose eccentricity is less than unity *i.e.*, $e < 1$.
- Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b$

  Eccentricity : $e=\sqrt{1-\frac{b^2}{a^2}} \Rightarrow a^2e^2 = a^2 - b^2 (0<e<1)$

  Foci : $S \equiv (ae, 0)$ and $S' \equiv (-ae, 0)$

  The equation of directrices are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$

  Vertices : $A' \equiv (-a, 0)$ and $A \equiv (a, 0)$



- The line segment $A'A$ of length $2a (a > b)$ is called the major axis of the ellipse.
- The y-axis intersects the ellipse at the points $B' \equiv (0, -b)$ and $B \equiv (0, b)$. The line segment $B'B$ of length $2b (b < a)$ is called the minor axis of the ellipse.
- The major and minor axis together are called principal axis of the ellipse.
- A chord which passes through focus is called a focal chord and which passes through the centre is called a diameter of the conic.
- A chord perpendicular to the major axis is called a double ordinate.
- The focal chord perpendicular to the major axis is called the latus rectum. Length of latus rectum (LL') $=\frac{2b^2}{a}=2a(1-e^2)$
- The sum of the focal distances of any point on the ellipse is equal to the major axis. Hence, distance of focus from the extremity of a minor axis is equal to semi major axis.

  i.e., $BS = CA$.
- The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>, <$ or $=0$ (respectively)
- A circle described on major axis as diameter is called the auxiliary circle. Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P and Q are called as the corresponding points on the ellipse and the auxiliary circle respectively 'θ' is called the eccentric angle of the point P on the ellipse $(0 \le \theta < 2\pi)$.

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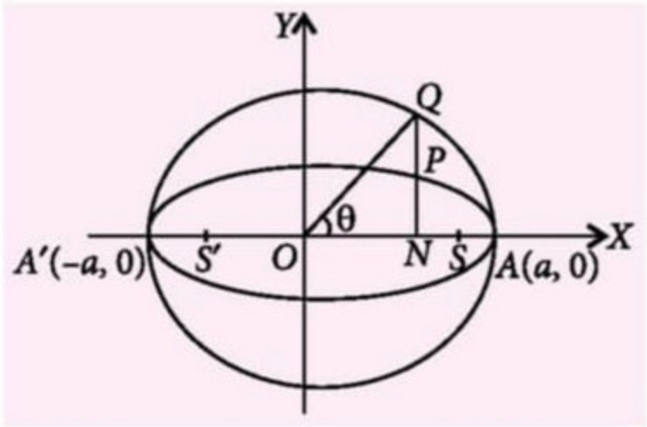


The equations $x = a\cos\theta$ and $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where θ is a parameter.

Note : If $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then, $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

- The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as c^2 is $<, =$ or $> a^2m^2 + b^2$.

  Hence, $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.

  The equation to the chord of the ellipse joining two points with eccentric angles α and β is given by $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$.

- $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is tangent to the ellipse at (x_1, y_1).

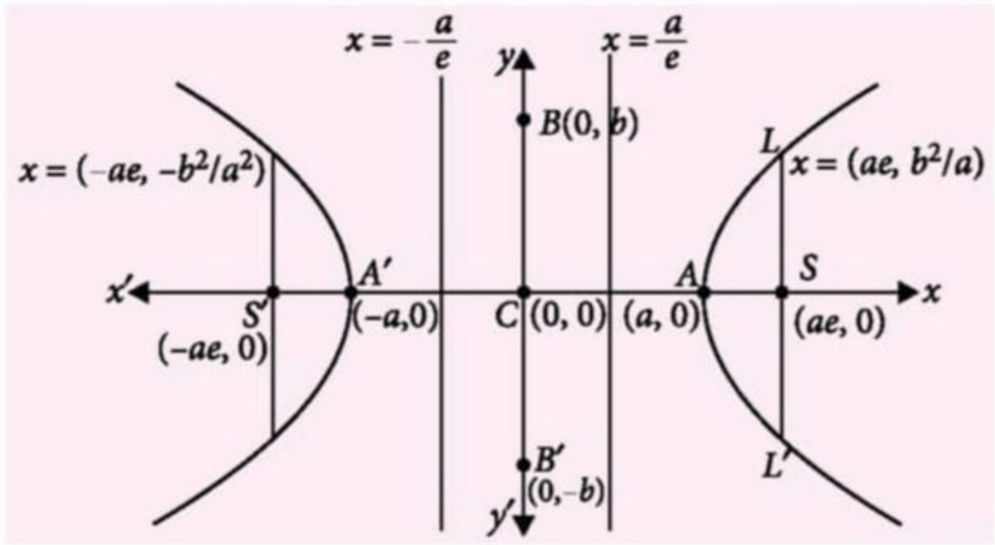
  For general ellipse replace x^2 by xx_1, y^2 by yy_1, $2x$ by $x + x_1$, $2y$ by $y + y_1$, $2xy$ by $xy_1 + yx_1$.
- $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse for all values of m.
- $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.
- The eccentric angles of point of contact of two parallel tangents differ by an angle π.
- Equation of the normal at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.
- Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ is $ax\cdot\sec\theta - by\cdot\operatorname{cosec}\theta = a^2 - b^2$.
- Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.
- Locus of the point of intersection of the tangents which meet at right angles is called the director circle. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ *i.e.* a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

HYPERBOLA

- The hyperbola is a conic whose eccentricity is greater than unity i.e., $e > 1$.
- Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

  Eccentricity : $e=\sqrt{1+\frac{b^2}{a^2}} \Rightarrow a^2e^2 = a^2 + b^2$

  Foci : $S \equiv (ae, 0)$ and $S' \equiv (-ae, 0)$.



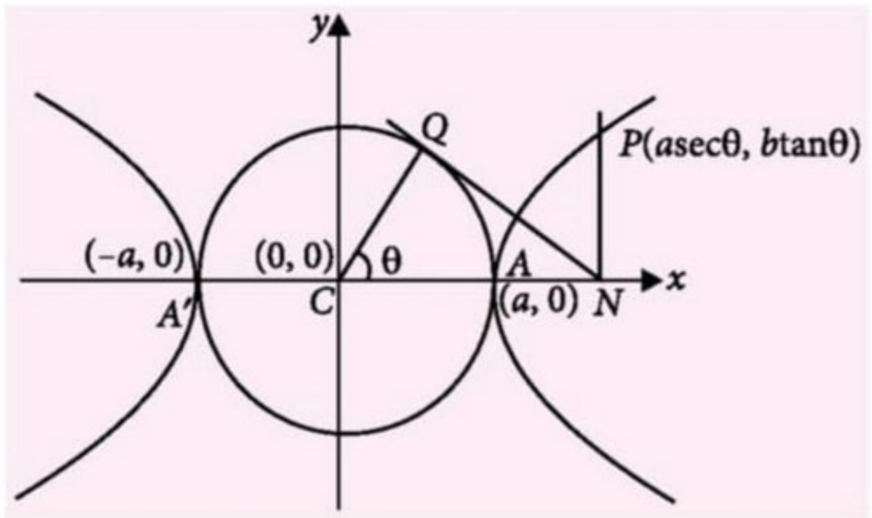
  Equation of directrices : $x=\frac{a}{e}$ and $x=-\frac{a}{e}$

  Vertices : $A \equiv (a, 0)$ and $A' \equiv (-a, 0)$.

  Latus rectum : $l=\frac{2b^2}{a}=2a(e^2-1)$.
- The line segment $A'A$ of length $2a$ in which the foci S' and S both lie is called the transverse axis of the hyperbola.
- The line segment $B'B$ between the two points $B' \equiv (0, -b)$ and $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola. The transverse axis and the conjugate axis of the hyperbola are together called the principal axes of the hyperbola.
- The difference of the focal distances of any point P on the hyperbola is constant and equal to transverse axis *i.e.* $||PS| - |PS'|| = 2a$. The distance SS' = focal length.
- Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called conjugate hyperbolas of each other .

  i.e. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbola of each other.

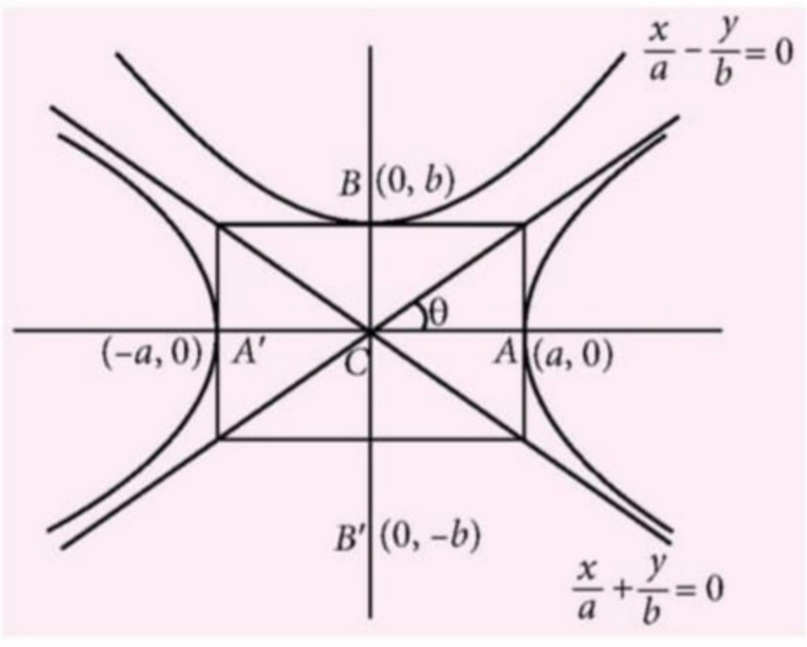
- If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate then $e_1^{-2} + e_2^{-2} = 1$.
- The particular kind of hyperbola in which the lengths of the transverse and conjugate axis are equal is called an equilateral hyperbola.
  Note : Eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of its latus rectum is equal to its transverse or conjugate axis.



- A circle drawn with centre C & transverse axis as a diameter is called the auxiliary circle of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$. Note from the figure that P and Q are called the "corresponding points" on the hyperbola and the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola $(0 \le \theta < 2\pi)$.
- The point (x_1, y_1) will be outside, on or inside the hyperbola according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 >, =$ or < 0 (respectively)
- The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $c^2 >, =, < a^2 m^2 - b^2$.
  Hence, $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$.
- Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.
  In general two tangents can be drawn from an external point (x_1, y_1) to the hyperbola and they are $y - y_1 = m_1 (x - x_1)$ and $y - y_2 = m_2 (x - x_2)$, where m_1 and m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$.
- Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$ is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$.

Also, $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

- Equation of a chord joining α and β is $\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$.
- The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $p(x_1, y_1)$ on it is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.
- The equation of the normal at the point $P(a \sec\theta, b \tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.
- The locus of the intersection of tangents which are at right angles is known as the director circle of the hyperbola. The equation to the director circle is $x^2 + y^2 = a^2 - b^2$.
  If $b^2 < a^2$ this circle is real; if $b^2 = a^2$ the radius of the circle is zero and it reduces to a point circle at the origin. In this case, the centre is the only point from which the tangents at right angles can be drawn to the curve. If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle and so no tangents at right angle can be drawn to the curve.
- Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
  Solving these two we get the quadratic as
  $(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0$ ...(i)



In order that $y = mx + c$ be an asymptote , both roots of equation (i) must approach infinity, the conditions for which are : coeff. of $x^2 = 0$ and coeff. of $x = 0$.

$\Rightarrow \quad b^2 - a^2m^2 = 0 \quad$ or $\quad m = \pm\dfrac{b}{a}$

and $a^2mc = 0 \quad \Rightarrow \quad c = 0.$

$\therefore$ Equations of asymptote are

$$\frac{x}{a}+\frac{y}{b}=0 \text{ and } \frac{x}{a}-\frac{y}{b}=0$$

On combining equation to the asymptotes, we get

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

- When $b = a$, then, the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ or $y = \pm x$ which are at right angles.
- If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.
- A hyperbola and its conjugate have the same asymptote.
- The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only.
- Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ .Then the point of intersection of $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$ gives the centre of the hyperbola.

Important Points on Rectangular Hyperbola

- Equation is $xy = c^2$ with parametric representation $x = ct, \ y = c/t, t \in R.$
- Equation of a chord joining the points $P(t_1)$ and $Q(t_2)$ is $x + t_1t_2\, y = c(t_1 + t_2)$.
- Equation of the tangent at $P(x_1, y_1)$ is $xy_1 + x_1y = 2c^2$ and at $P(ct, c/t)$ is $\dfrac{x}{t}+ty=2c.$
- Chord with a given middle point as (h, k) is $kx + hy = 2hk$.
- Equation of the normal at $P(ct, c/t)$ is $xt^3 - yt = c(t^4 - 1)$.
- Vertex of this hyperbola is (c, c) and $(-c, -c)$, focus is $(\sqrt{2}c\,, \sqrt{2}c)$ and $(-\sqrt{2}c, -\sqrt{2}c)$, the directrices are $x + y = \pm\sqrt{2}\,c$ and latus rectum $= 2\sqrt{2}c.$

PROBLEMS

Single Correct Answer Type

1. The eccentricity of the hyperbola conjugate to $x^2 - 3y^2 = 2x + 8$ is

(a) $\dfrac{2}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) 2 (d) none of these

2. The pole of the straight line $x + 4y = 4$ with respect to ellipse $x^2 + 4y^2 = 4$ is

(a) (1, 4) (b) (1, 1) (c) (4, 1) (d) (4, 4)

3. In the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,$ the equation of diameter conjugate to the diameter $y=\dfrac{b}{a}x,$ is

(a) $y=-\dfrac{b}{a}x$ (b) $y=-\dfrac{a}{b}x$

(c) $x=-\dfrac{b}{a}y$ (d) none of these

4. Minimum area of the triangle by any tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with the coordinate axes is

(a) $\dfrac{a^2+b^2}{2}$ (b) $\dfrac{(a+b)^2}{2}$

(c) ab (d) $\dfrac{(a-b)^2}{2}$

5. The eccentricity of the ellipse $25x^2 + 16y^2 - 150x - 175 = 0$ is

(a) 2/5 (b) 2/3

(c) 4/5 (d) 3/5

6. If $5x^2+\lambda y^2=20$ represents a rectangular hyperbola, then λ equals

(a) 5 (b) 4

(c) −5 (d) none of these

7. The locus of the point of intersection of the lines $bxt - ayt = ab$ and $bx + ay = abt$ is

(a) a parabola (b) an ellipse

(c) a hyperbola (d) none of these

8. The eccentricity of the hyperbola can never be equal to

(a) $\sqrt{\dfrac{9}{5}}$ (b) $2\sqrt{\dfrac{1}{9}}$ (c) $3\sqrt{\dfrac{1}{8}}$ (d) 2

9. The locus of the centre of a circle, which touches externally the given two circles, is

(a) circle (b) parabola

(c) hyperbola (d) ellipse

10. The foci of the hyperbola $9x^2 - 16y^2 = 144$ are

(a) $(\pm4, 0)$ (b) $(0, \pm4)$

(c) $(\pm5, 0)$ (d) $(0, \pm5)$

11. The equation $x^2 + 4xy + y^2 + 2x + 4y + 2 = 0$ represents

(a) an ellipse (b) a pair of straight lines

(c) a hyperbola (d) none of these

12. The auxiliary equation of circle of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is

(a) $x^2+y^2=a^2$ (b) $x^2+y^2=b^2$

(c) $x^2+y^2=a^2+b^2$ (d) $x^2+y^2=a^2-b^2$

13. If the line $y=2x+\lambda$ be a tangent to the hyperbola $36x^2-25y^2=3600$, then $\lambda=$

(a) 16 (b) -16

(c) ±16 (d) none of these

14. The equation of the tangents to the conic $3x^2-y^2=3$ perpendicular to the line $x+3y=2$ is

(a) $y=3x\pm\sqrt{6}$ (b) $y=6x\pm\sqrt{3}$

(c) $y=x\pm\sqrt{6}$ (d) $y=3x\pm6$

15. The locus of the point of intersection of any two perpendicular tangents to the hyperbola is a circle which is called the director circle of the hyperbola, then the equation of this circle is

(a) $x^2+y^2=a^2+b^2$ (b) $x^2+y^2=a^2-b^2$

(c) $x^2+y^2=2ab$ (d) none of these

16. The equation of the tangents to the hyperbola $3x^2-4y^2=12$ which cuts equal intercepts from the axes, are

(a) $y+x=\pm1$ (b) $y-x=\pm1$

(c) $3x+4y=\pm1$ (d) $3x-4y=\pm1$

17. If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which pass through the point (6, 2), then

(a) $m_1+m_2=\frac{24}{11}$ (b) $m_1m_2=\frac{2}{11}$

(c) $m_1+m_2=\frac{48}{11}$ (d) $m_1m_2=\frac{11}{20}$

18. The value of m for which $y=mx+6$ is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$, is

(a) $\sqrt{\frac{17}{20}}$ (b) $\sqrt{\frac{20}{17}}$

(c) $\sqrt{\frac{3}{20}}$ (d) $\sqrt{\frac{20}{3}}$

19. The equation of the tangent to the conic $x^2-y^2-8x+2y+11=0$ at (2, 1) is

(a) $x+2=0$ (b) $2x+1=0$

(c) $x-2=0$ (d) $x+y+1=0$

20. If the straight line $x\cos\alpha+y\sin\alpha=p$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

(a) $a^2\cos^2\alpha+b^2\sin^2\alpha=p^2$

(b) $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$

(c) $a^2\sin^2\alpha+b^2\cos^2\alpha=p^2$

(d) $a^2\sin^2\alpha-b^2\cos^2\alpha=p^2$

21. If the tangent on the point $(2\sec\phi, 3\tan\phi)$ of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to $3x-y+4=0$, then the value of ϕ is

(a) 45° (b) 60° (c) 30° (d) 75°

22. What is the slope of the tangent line drawn to the hyperbola $xy=a(a\neq0)$ at the point $(a, 1)$?

(a) $1/a$ (b) $-1/a$ (c) a (d) $-a$

23. The straight line $x+y=\sqrt{2}p$ will touch the hyperbola $4x^2-9y^2=36$, if

(a) $p^2=2$ (b) $p^2=5$

(c) $5p^2=2$ (d) $2p^2=5$

24. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then

(a) Q lies inside C but outside E

(b) Q lies outside both C and E

(c) P lies inside both C and E

(d) P lies inside C but outside E

25. The distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$ is

(a) 8 (b) 12 (c) 18 (d) 24

26. The equation of the ellipse whose vertices are $(\pm5, 0)$ and foci are $(\pm4, 0)$ is

(a) $9x^2+25y^2=225$ (b) $25x^2+9y^2=225$

(c) $3x^2+4y^2=192$ (d) none of these

27. The equation of the ellipse whose one of the vertices is (0, 7) and the corresponding directrix is $y=12$, is

(a) $95x^2+144y^2=4655$ (b) $144x^2+95y^2=4655$

(c) $95x^2+144y^2=13680$ (d) none of these

28. For the ellipse $3x^2+4y^2=12$, the length of latus rectum is

(a) $\frac{3}{2}$ (b) 3 (c) $\frac{8}{3}$ (d) $\sqrt{\frac{3}{2}}$

29. Eccentricity of the ellipse $9x^2 + 25y^2 = 225$ is

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{9}{25}$ (d) $\frac{\sqrt{34}}{5}$

30. What is the equation of the ellipse with foci $(\pm 2, 0)$ and eccentricity $= \frac{1}{2}$?

(a) $3x^2 + 4y^2 = 48$ (b) $4x^2 + 3y^2 = 48$
(c) $3x^2 + 4y^2 = 0$ (d) $4x^2 + 3y^2 = 0$

31. The equation of ellipse whose distance between the foci is equal to 8 and distance between the directrices is 18, is

(a) $5x^2 - 9y^2 = 180$ (b) $9x^2 + 5y^2 = 180$
(c) $x^2 + 9y^2 = 180$ (d) $5x^2 + 9y^2 = 180$

32. The centre of the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$ is

(a) (1,3) (b) (2, 3)
(c) (3, 2) (d) (3, 1)

33. The equation of an ellipse whose eccentricity is 1/2 and the vertices are (4, 0) and (10, 0) is

(a) $3x^2 + 4y^2 - 42x + 120 = 0$
(b) $3x^2 + 4y^2 + 42x + 120 = 0$
(c) $3x^2 + 4y^2 + 42x - 120 = 0$
(d) $3x^2 + 4y^2 - 42x - 120 = 0$

34. The eccentricity of the ellipse $9x^2 + 5y^2 - 18x - 20y - 16 = 0$ is

(a) 1/2 (b) 2/3 (c) 1/3 (d) 3/4

35. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

(a) outside the ellipse (b) on the ellipse
(c) on the major axis (d) on the minor axis

36. The locus of the point of intersection of mutually perpendicular tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

(a) a straight line (b) a parabola
(c) a director circle (d) none of these

37. The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$, are

(a) $y = \pm 3$ (b) $x = \pm\sqrt{5}$
(c) $y = 0, y = 6$ (d) none of these

38. The line $y = mx + c$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $c =$

(a) $-(2am + bm^2)$ (b) $\frac{(a^2 + b^2)m}{\sqrt{a^2 + b^2m^2}}$

(c) $-\frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$ (d) $\frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2}}$

39. The equation of the normal at the point (2, 3) on the ellipse $9x^2 + 16y^2 = 180$, is

(a) $3y = 8x - 10$ (b) $3y - 8x + 7 = 0$
(c) $8y + 3x + 7 = 0$ (d) $3x + 2y + 7 = 0$

40. If (4, 0) and (–4, 0) be the vertices and (6, 0) and (–6, 0) be the foci of a hyperbola, then its eccentricity is

(a) 5/2 (b) 2 (c) 3/2 (d) $\sqrt{2}$

41. If $(0, \pm 4)$ and $(0, \pm 2)$ be the foci and vertices of a hyperbola, then its equation is

(a) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (b) $\frac{x^2}{12} - \frac{y^2}{4} = 1$
(c) $\frac{y^2}{4} - \frac{x^2}{12} = 1$ (d) $\frac{y^2}{12} - \frac{x^2}{4} = 1$

42. The eccentricity of the hyperbola $4x^2 - 9y^2 = 16$, is

(a) $\frac{8}{3}$ (b) $\frac{5}{4}$ (c) $\frac{\sqrt{13}}{3}$ (d) $\frac{4}{3}$

43. The eccentricity of the hyperbola $2x^2 - y^2 = 6$ is

(a) $\sqrt{2}$ (b) 2 (c) 3 (d) $\sqrt{3}$

44. The equation $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$ represents

(a) parabola (b) ellipse
(c) hyperbola (d) two straight lines

45. The equation of the hyperbola referred to its axes as axes of coordinate and whose distance between the foci is 16 and eccentricity is $\sqrt{2}$, is

(a) $x^2 - y^2 = 16$ (b) $x^2 - y^2 = 32$
(c) $x^2 - 2y^2 = 16$ (d) $y^2 - x^2 = 16$

46. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

(a) 1 (b) 5 (c) 7 (d) 9

47. The eccentricity of the hyperbola $\frac{\sqrt{1999}}{3}(x^2 - y^2) = 1$ is

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true, Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

48. **Statement-1 :** The foci of the hyperbola $xy = 36$ are $(6\sqrt{2}, 6\sqrt{2})$ and $(-6\sqrt{2}, -6\sqrt{2})$.

Statement-2 : The foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\left(\pm\sqrt{a^2 - b^2}, 0\right)$.

49. **Statement-1 :** The angle of intersection between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = ab$ is $\tan^{-1}\left(\frac{b-a}{\sqrt{ab}}\right)$.

Statement-2 : The point of intersection of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = ab$ is $\left(\sqrt{\frac{ab}{a+b}}, \sqrt{\frac{ab}{a+b}}\right)$.

50. **Statement-1 :** The condition on a and b for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through $(a, -b)$ are bisected by $x + y = b$ is $a^2 + 6ab - 7b^2 > 0$.

Statement 2 : Equation of the chord of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose mid point is (x_1, y_1) of the form $T = S_1$ *i.e.* $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$.

51. **Statement-1 :** If the point (x, y) lies on the curve $2x^2 + y^2 - 24y + 80 = 0$ then the maximum value of $x^2 + y^2$ is 400.

Statement-2 : The point (x, y) is at a distance of $\sqrt{x^2 + y^2}$ from origin.

Comprehension Type

Paragraph for Q.No. 52 - 54

A parabola is drawn through two given points $A(1, 0)$ and $B(-1, 0)$ such that its directrix always touches the circle $x^2 + y^2 = 4$. Then

52. The equation of directrix is of the form

(a) $x\cos\alpha + y\sin\alpha = 1$ (b) $x\cos\alpha + y\sin\alpha = 2$

(c) $x\cos\alpha + y\sin\alpha = 3$ (d) $x\tan\alpha + y\sec\alpha = 2$

53. The locus of focus of the parabola is

(a) $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (b) $\frac{x^2}{4} + \frac{y^2}{5} = 1$

(c) $\frac{x^2}{3} + \frac{y^2}{4} = 1$ (d) $\frac{x^2}{5} + \frac{y^2}{4} = 1$

54. The maximum possible length of semi latus rectum is

(a) $2 + \sqrt{3}$ (b) $3 + \sqrt{3}$

(c) $4 + \sqrt{3}$ (d) $1 + \sqrt{3}$

Paragraph for Q.No. 55 - 57

Let P, Q, R be three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let P', Q', R' be their corresponding points on it's auxiliary circle, then

55. The maximum area of the triangle PQR is

(a) $\frac{3\sqrt{3}}{4}ab$ (b) $\frac{3\sqrt{3}}{2}ab$

(c) $\frac{\sqrt{3}}{4}ab$ (d) πab

56. $\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta P'Q'R'} =$

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$

(c) $\frac{1}{2}$ (d) none of these

57. When the area of triangle PQR is maximum, the centroid of triangle $P'Q'R'$ lies at

(a) one focus (b) one vertex

(c) centre (d) on one directrix

Paragraph for Q.No. 58 - 60

Suppose an ellipse and a circle are respectively given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i) and

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(ii)

The equation,

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda(x^2 + y^2 + 2gx + 2fy + c) = 0 \quad \text{...(iii)}$$

represents a curve which passes through the common points of the ellipse (i) and the circle (ii).

We can choose λ so that the equation (iii) represents a pair of straight lines. In general we get three values of λ, indicating three pair of straight lines can be drawn through the points. Also when (iii) represents a pair of straight lines they are parallel to the lines $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \lambda(x^2 + y^2) = 0$, which represents a pair of lines equally inclined to axes (the term containing y is absent). Hence two straight lines through the points

of intersection of an ellipse and any circle make equal angles with the axes.

58. The radius of the circle passing through the point of intersection of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x^2-y^2=0$ is

(a) $\frac{ab}{\sqrt{a^2+b^2}}$ (b) $\frac{\sqrt{2}\,ab}{\sqrt{a^2+b^2}}$

(c) $\frac{a^2-b^2}{\sqrt{a^2+b^2}}$ (d) $\frac{a^2+b^2}{\sqrt{a^2+b^2}}$

59. Suppose two lines are drawn through the common points of intersection of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $x^2+y^2+2gx+2fy+c=0$. If these lines are inclined at an angle α, β to x-axis then

(a) $\alpha=\beta$ (b) $\alpha+\beta=\frac{\pi}{2}$

(c) $\alpha+\beta=\pi$ (d) $\alpha+\beta=2\tan^{-1}\left(\frac{b}{a}\right)$

60. The no. of pair of straight lines through the points of intersection of $x^2-y^2=1$ and $x^2+y^2-4x-5=0$ is

(a) 0 (b) 1

(c) 2 (d) 3

Matrix–Match Type

61. Match the following.

Column-I		Column-II	
(A)	The locus of mid-points of chords of an ellipse which are drawn through an end of minor axis, is	(p)	hyperbola
(B)	The locus of an end of latus rectum of all ellipses having a given major axis is	(q)	circle
(C)	The locus of the foot of perpendicular from a focus of the ellipse on any tangent is	(r)	parabola
(D)	A variable line is drawn through a fixed point cuts axes at A and B. The locus of the mid point of AB is	(s)	ellipse

Integer Answer Type

62. If e is the eccentricity of the hyperbola $(5x-10)^2+(5y+15)^2=(12x-5y+1)^2$ then $\frac{25e}{13}$ is equal to

63. The minimum distance of $4x^2+y^2+4x-4y+5=0$ from the line $-4x+3y=3$ is

64. The distance between the directrices of the ellipse $(4x-8)^2+16y^2=(x+\sqrt{3}y+10)^2$ is K then $\frac{K}{2}$ is

65. Number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

66. If L be the length of common tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$ and the circle $x^2+y^2=16$ intercepted by the coordinate axis then $\frac{\sqrt{3}L}{2}$ is

SOLUTIONS

1. (c): Given, equation of hyperbola is $x^2-3y^2=2x+8$

$\Rightarrow x^2-2x-3y^2=8$

$\Rightarrow (x-1)^2-3y^2=9 \Rightarrow \frac{(x-1)^2}{9}-\frac{y^2}{3}=1$

Conjugate of this hyperbola is $-\frac{(x-1)^2}{9}+\frac{y^2}{3}=1$

and its eccentricity $(e)=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$

Here, $a^2=9, b^2=3 \therefore e=\sqrt{\frac{9+3}{3}}=2$

2. (b) : We know that equation of polar at point (h, k) is

$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \Rightarrow \frac{hx}{4}+\frac{ky}{1}=1 \Rightarrow hx+4ky=4$ ...(i)

which is similar to given straight line $x+4y=4$ ...(ii)

Comparing (i) and (ii), we get $h=1, k=1$.

Hence, the point is (1, 1).

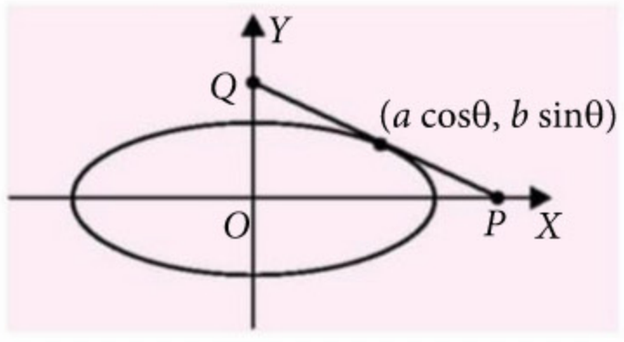
3. (a) : $y=\frac{-b}{a}x$ (Two diameters $y=m_1x$ and $y=m_2x$ will be conjugate diameters, if $m_1m_2=-\frac{b^2}{a^2}$).

4. (c) : Equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

$P=\left(\frac{a}{\cos\theta}, 0\right)$

$Q=\left(0, \frac{b}{\sin\theta}\right)$



Area of $\Delta OPQ = \frac{1}{2}\left|\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right)\right| = \frac{ab}{|\sin 2\theta|}$

$\therefore$ $(\text{Area})_{\min} = ab$.

5. (d) : $25(x-3)^2 + 16y^2 = 400$

$\frac{(x-3)^2}{16} + \frac{y^2}{25} = 1 \quad \therefore \quad e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

6. (c) : Since the general equation of second degree represents a rectangular hyperbola, if $\Delta \neq 0$, $h^2 > ab$ and coefficient of x^2 + coefficient of $y^2 = 0$. Therefore the given equation represents a rectangular hyperbola, if $\lambda + 5 = 0$ *i.e.*, $\lambda = -5$.

7. (c) : Multiplying both, we get

$(bx)^2 - (ay)^2 = (ab)^2 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

which is the standard equation of hyperbola.

8. (b) : Since $e > 1$ always for hyperbola and $\frac{2}{3} < 1$.

9. (c) : We know that when a circle touches externally to the two given circles, then the locus of the circle will be hyperbola.

10. (c) : The equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Now $b^2 = a^2(e^2 - 1) \Rightarrow e = \frac{5}{4}$

Hence foci are $(\pm ae, 0) \Rightarrow \left(\pm 4 \cdot \frac{5}{4}, 0\right)$ *i.e.*, $(\pm 5, 0)$.

11. (c) : Since $h^2 > ab$

Hence it is a hyperbola.

12. (a) : The equation is $(x-0)^2 + (y-0)^2 = a^2$.

13. (c) : If $y = 2x + \lambda$ is tangent to given hyperbola, then

$\lambda = \pm\sqrt{a^2m^2 - b^2} = \pm\sqrt{(100)(4) - 144} = \pm\sqrt{256} = \pm 16$

14. (a) : Tangents to $\frac{x^2}{1} - \frac{y^2}{3} = 1$ and perpendicular to $x + 3y - 2 = 0$ are given by $y = 3x \pm \sqrt{9 - 3} = 3x \pm \sqrt{6}$.

15. (b) : Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Tangents to hyperbola are $y = mx \pm \sqrt{a^2m^2 - b^2}$ ...(i)

Tangents perpendicular to (i) are $y = \frac{-1}{m}x \pm \sqrt{\frac{a^2}{m^2} - b^2}$

Eliminating m, we get $x^2 + y^2 = a^2 - b^2$.

16. (b) : The tangent at (h, k) is $\frac{x}{4/h} - \frac{y}{3/k} = 1$

$\therefore \quad \frac{4}{h} = \frac{3}{k} \Rightarrow \frac{h}{k} = \frac{4}{3}$ ...(i)

$3h^2 - 4k^2 = 12$ ...(ii)

Using (i) and (ii), we get the tangents as $y - x = \pm 1$.

17. (a) : The line through (6, 2) is

$y - 2 = m(x - 6) \Rightarrow y = mx + 2 - 6m$

Now from condition of tangency, $(2 - 6m)^2 = 25m^2 - 16$

$\Rightarrow \quad 36m^2 + 4 - 24m - 25m^2 + 16 = 0$

$\Rightarrow \quad 11m^2 - 24m + 20 = 0$

Obviously its roots are m_1 and m_2, therefore

$m_1 + m_2 = \frac{24}{11}$ and $m_1 m_2 = \frac{20}{11}$

18. (a) : If $y = mx + c$ touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$.

Here, $c = 6$, $a^2 = 100$, $b^2 = 49$

$\therefore \quad 36 = 100m^2 - 49 \Rightarrow 100m^2 = 85 \Rightarrow m = \sqrt{\frac{17}{20}}$.

19. (c) : Equation of the tangent to $x^2 - y^2 - 8x + 2y + 11 = 0$ at $(2, 1)$ is $2x - y - 4(x+2) + (y+1) + 11 = 0$ or $x = 2$.

20. (b) : $x\cos\alpha + y\sin\alpha = p \Rightarrow y = -\cot\alpha\, x + p\,\text{cosec}\alpha$

It is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Therefore, $p^2\,\text{cosec}^2\alpha = a^2\cot^2\alpha - b^2$

$\Rightarrow \quad a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

21. (c) : We have, $x = 2\sec\phi$ and $y = 3\tan\phi$

$\Rightarrow \quad \frac{dx}{d\phi} = 2\sec\phi\tan\phi$ and $\frac{dy}{d\phi} = 3\sec^2\phi$

$\therefore \quad \frac{dy}{dx} = \frac{dy/d\phi}{dx/d\phi} = \frac{3\sec^2\phi}{2\sec\phi\tan\phi}$

$\frac{dy}{dx} = \frac{3}{2}\text{cosec}\phi$ ...(i)

But, tangent is parallel to $3x - y + 4 = 0$ ...(ii)

By (i) and (ii), $\frac{3}{2}\text{cosec}\phi = 3 \Rightarrow \text{cosec}\phi = 2 \therefore \phi = 30°$.

22. (b) : Given equation of hyperbola is $xy = a$

Slope of tangent at point (x_1, y_1) is

$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \quad \therefore \quad \frac{xdy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

At point $(a, 1)$, we have $m = \left(\frac{dy}{dx}\right)_{(a,1)} = -\frac{1}{a}$

23. (d) : The condition for the line $y = mx + c$ will touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$

Here, $m=-1, c=\sqrt{2}p,\ a^2=9, b^2=4$

$\therefore$ We get $2p^2 = 5$

24. (d) : The given ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$. The value of the expression $\frac{x^2}{9}+\frac{y^2}{4}-1$ is positive for $x = 1, y = 2$ and negative for $x = 2, y = 1$. Therefore P lies outside E and Q lies inside E. The value of the expression $x^2 + y^2 - 9$ is negative for both the points P and Q. Therefore P and Q both lie inside C. Hence, P lies inside C but outside E.

25. (c) : We have, $a=6, b=2\sqrt{5}$

$$\therefore \quad b^2=a^2(1-e^2)\Rightarrow \frac{20}{36}=(1-e^2)\Rightarrow e=\sqrt{\frac{16}{36}}=\frac{2}{3}$$

But directrices are $x=\pm\frac{a}{e}$

Hence, distance between them is $2\cdot\frac{6}{2/3}=18.$

26. (a) : Vertices $\equiv (\pm5, 0) \equiv (\pm a, 0) \Rightarrow a = 5$

Foci $(\pm4,0)\equiv(\pm ae,0) \Rightarrow e=\frac{4}{5}$

$$\therefore \quad b^2=(25)\left(1-\frac{16}{25}\right)=9 \Rightarrow b = 9$$

Hence equation is $\frac{x^2}{25}+\frac{y^2}{9}=1$ *i.e.*, $9x^2 + 25y^2 = 225$

27. (b) : We have, vertex $= (0,7) \Rightarrow b = 7$

Directrix $y = 12 \Rightarrow \frac{b}{e}=12 \Rightarrow e=\frac{7}{12}$

Also, $a=7\sqrt{\frac{95}{144}}=\frac{7\sqrt{95}}{12}$

Hence equation of ellipse is $144x^2 + 95y^2 = 4655.$

28. (b) : $\frac{x^2}{4}+\frac{y^2}{3}=1.$ Latus rectum $=\frac{2b^2}{a}=3.$

29. (b) : The ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$

$$\therefore \quad e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$$

30. (a) : Since, $ae=\pm2 \Rightarrow a=\pm4 \quad (\because e=1/2)$

Now $b^2 = a^2(1 - e^2) \Rightarrow b^2 = 16(1 - 1/4) \Rightarrow b^2 = 12$

Hence ellipse is $\frac{x^2}{16}+\frac{y^2}{12}=1 \Rightarrow 3x^2+4y^2=48$

31. (d) : We have, $2ae=8, \frac{2a}{e}=18 \Rightarrow a=\sqrt{4\times9}=6$

$$e=\frac{2}{3},\ b=6\sqrt{1-\frac{4}{9}}=\frac{6}{3}\sqrt{5}=2\sqrt{5}$$

Hence the required equation is $\frac{x^2}{36}+\frac{y^2}{20}=1$

i.e., $5x^2 + 9y^2 = 180.$

32. (b) : $4(x - 2)^2 + 9(y - 3)^2 = 36$

Hence, the centre is (2, 3).

33. (a) : Major axis $= 6 = 2a \Rightarrow a = 3$

Now, $e=\frac{1}{2} \Rightarrow b=3\sqrt{1-\frac{1}{4}}=\frac{3\sqrt{3}}{2}$

Also centre is (7, 0)

Equation is $\frac{(x-7)^2}{9}+\frac{y^2}{(27/4)}=1$

$\Rightarrow \quad 3x^2 + 4y^2 - 42x + 120 = 0$

34. (b) : Given equation can be written as

$$\frac{(x-1)^2}{5}+\frac{(y-2)^2}{9}=1 \ \therefore\ e=\sqrt{\frac{b^2-a^2}{b^2}}=\sqrt{\frac{9-5}{9}}=\frac{2}{3}$$

35. (c) : $E \equiv 4 + 9(3)^2 - 16(1) - 54(3) + 61 < 0$

Therefore, the point is inside the ellipse.

$$\frac{4(x-2)^2}{36}+\frac{9(y-3)^2}{36}=1$$

Equation of major axis is $y - 3 = 0$ and point (1, 3) lies on it.

36. (c) : It is the director circle.

37. (c) : Change the equation $9x^2 + 5y^2 - 30y = 0$ in standard form $9x^2 + 5(y^2 - 6y) = 0$

$$\Rightarrow 9x^2+5(y^2-6y+9)=45\Rightarrow \frac{x^2}{5}+\frac{(y-3)^2}{9}=1$$

$\because \quad a^2 < b^2$, so axis of ellipse is on y-axis.

At y axis, put $x = 0$, so we get

$0 + 5y^2 - 30y = 0 \quad \Rightarrow \quad y = 0, y = 6$

i.e., tangents at vertex is $y = 0, y = 6.$

38. (c) : As we know that the line $lx + my + n = 0$ is normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$. But in this condition, we have to replace l by m, m by -1 and n by c, then the required condition is $c=\pm\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}.$

39. (b) : $\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$, which is the equation of normal at point (x_1, y_1).

In the given ellipse, $a^2 = 20, b^2 = \frac{180}{16}$

Hence the equation of normal at the point (2, 3) is

$$\frac{x-2}{2/20} = \frac{y-3}{48/180} \Rightarrow 40(x-2) = 15(y-3)$$

$\Rightarrow \quad 8x - 3y = 7 \Rightarrow 3y - 8x + 7 = 0.$

40. (c) : Vertices $(\pm 4, 0) \equiv (\pm a, 0) \Rightarrow a = 4$

Foci $(\pm 6, 0) \equiv (\pm ae, 0) \quad \Rightarrow \quad e = \frac{6}{4} = \frac{3}{2}$

41. (c) : Foci $= (0, \pm 4) \equiv (0, \pm be) \Rightarrow be = 4$

Vertices $(=0, \pm 2) \equiv (0, \pm b) \Rightarrow b = 2 \Rightarrow a = 2\sqrt{3}$

Hence equation is $\frac{-x^2}{(2\sqrt{3})^2} + \frac{y^2}{(2)^2} = 1$ or $\frac{y^2}{4} - \frac{x^2}{12} = 1$

42. (c) : Given equation of hyperbola, $\frac{x^2}{4} - \frac{y^2}{(16/9)} = 1,$

$\therefore \quad a = 2, b = \frac{4}{3}$. As we know, $b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad \frac{16}{9} = 4(e^2 - 1) \Rightarrow e^2 = \frac{13}{9} \quad \therefore \quad e = \frac{\sqrt{13}}{3}$

43. (d) : $\frac{x^2}{(6/2)} - \frac{y^2}{6} = 1 \Rightarrow a^2 = 3$ and $b^2 = 6$

Therefore $e = \sqrt{\frac{b^2}{a^2} + 1} \Rightarrow e = \sqrt{3}$

44. (c) : $\Delta \neq 0, h^2 > ab$

45. (b) : $2ae = 16, e = \sqrt{2} \Rightarrow a = 4\sqrt{2}$ and $b = 4\sqrt{2}$

$\therefore$ Equation is $\frac{x^2}{(4\sqrt{2})^2} - \frac{y^2}{(4\sqrt{2})^2} = 1 \Rightarrow x^2 - y^2 = 32$

46. (c) : Hyperbola is $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

$$a = \sqrt{\frac{144}{25}}, b = \sqrt{\frac{81}{25}}, e_1 = \sqrt{1 + \frac{81}{144}} = \sqrt{\frac{225}{144}} = \frac{15}{12} = \frac{5}{4}$$

Therefore, foci $= (\pm ae_1, 0) = \left(\pm \frac{12}{5} \cdot \frac{5}{4}, 0\right) = (\pm 3, 0)$

Also, focus of ellipse = $(4e, 0)$.

$\Rightarrow \quad e = \frac{3}{4}$. Hence, $b^2 = 16\left(1 - \frac{9}{16}\right) = 7$

47. (b) : Here $a = b$, so it is a rectangular hyperbola.

Hence, eccentricity, $e = \sqrt{2}$.

48. (c) : Foci of $xy = 36$ are $(c\sqrt{2}, c\sqrt{2})$

and $(-c\sqrt{2}, -c\sqrt{2}) = (6\sqrt{2}, 6\sqrt{2}), (-6\sqrt{2}, -6\sqrt{2})$

$\therefore$ Statement-1 is true.

The foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\left(\pm\sqrt{a^2 + b^2}, 0\right)$

$\therefore$ Statement-2 is false.

49. (c) : The point of intersection is, $\left(\sqrt{\frac{a^2 b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$,

with $m_1 = \frac{-b^2}{a^2}\sqrt{\frac{a}{b}}, m_2 = -\sqrt{\frac{a}{b}}.$

$$\Rightarrow \theta = \tan^{-1}\left|\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)\right| = \tan^{-1}\left(\frac{b-a}{\sqrt{ab}}\right).$$

50. (a) : Let the mid point $(t, b - t)$.

$$\frac{tx}{2a^2} + \frac{(b-t)y}{2b^2} = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2}$$

It passes through $(a, -b)$.

$$\frac{ta}{2a^2} - \frac{b(b-t)}{2b^2} = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2}$$

$\Rightarrow \quad t^2(a^2 + b^2) - ab(3a + b)t + 2a^2b^2 = 0$

For real t, $a^2b^2(3a + b)^2 - 4(a^2 + b^2)2a^2b^2 > 0$

$9a^2 + 6ab + b^2 - 8a^2 - 8b^2 > 0$

$a^2 + 6ab - 7b^2 > 0$

51. (a) : Given equation of ellipse is

$\frac{x^2}{32} + \frac{(y-12)^2}{64} = 1$. The maximum value of $\sqrt{x^2 + y^2}$ is the distance between (0, 0) & (0, 20).

52. (b) **53. (a)** **54. (a)**

52 - 54 : Any point on circle $x^2 + y^2 = 4$ is $(2\cos\alpha, 2\sin\alpha)$

$\therefore$ Equation of directrix is $x(\cos\alpha) + y(\sin\alpha) - 2 = 0$

Let focus be (x_1, y_1). Then as $A(1, 0)$, $B(-1, 0)$ lie on parabola we must have

$$\left.\begin{aligned}(x_1 - 1)^2 + y_1^2 &= (\cos\alpha - 2)^2 \\ (x_1 + 1)^2 + y_1^2 &= (\cos\alpha + 2)^2\end{aligned}\right\} \Rightarrow x_1 = 2\cos\alpha, \quad y_1 = \pm\sqrt{3}\sin\alpha$$

$\therefore$ Locus of focus is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ and focus is of the form $(2\cos\alpha, \pm\sqrt{3}\sin\alpha)$.

$\therefore$ Length of semi latus rectum of parabola

$= \perp^r$ distance from focus to directrix $\left|2 \pm \sqrt{3}\right| \sin^2\alpha.$

Hence maximum possible length $= 2 + \sqrt{3}$

55. (a) : Let $P = (a\cos\alpha, b\sin\alpha)$, $P' = (a\cos\alpha, a\sin\alpha)$
$Q = (a\cos\beta, b\sin\beta)$, $Q' = (a\cos\beta, a\sin\beta)$
$R = (a\cos\gamma, b\sin\gamma)$, $R' = (a\cos\gamma, a\sin\gamma)$.

Area of ΔPQR is $2ab\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}$

Its max value is $2ab\cdot\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{3}/2\right)\left(\sqrt{3}/2\right)=\frac{3\sqrt{3}ab}{4}$

56. (b) : $\dfrac{\text{Area of } \Delta PQR}{\text{Area of } \Delta P'Q'R'}$

$$=\frac{\frac{1}{2}ab\begin{vmatrix}\cos\alpha-\cos\gamma & \sin\alpha-\sin\gamma\\ \cos\alpha-\cos\beta & \sin\alpha-\sin\beta\end{vmatrix}}{\frac{1}{2}a^2\begin{vmatrix}\cos\alpha-\cos\gamma & \sin\alpha-\sin\gamma\\ \cos\alpha-\cos\beta & \sin\alpha-\sin\beta\end{vmatrix}}=\frac{b}{a}$$

57. (c) : Area of ΔPQR is max when $\alpha-\beta=\beta-\gamma=\gamma-\alpha=120°$ and $\Delta P'Q'R'$ is equilateral hence its centroid is (0, 0) centre of the ellipse.

58. (b) : $x^2+y^2=\dfrac{2a^2b^2}{a^2+b^2}$

$\therefore$ Radius of the circle $=\sqrt{\dfrac{2a^2b^2}{a^2+b^2}}=\dfrac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$

59. (c) : As the lines joining common point of intersection must be equally inclined to the axis

$\tan\alpha = -\tan\beta \Rightarrow \alpha+\beta = n\pi$

60. (c) : Any curve through their point of intersection is $x^2+y^2-4x-5+\lambda(x^2-y^2-1)=0$

$\Rightarrow (1+\lambda)x^2+(1-\lambda)y^2-4x-5-\lambda=0$

Curve will be pair of straight lines, if

$abc+2fgh-af^2-by^2-ch^2=0$

$\Rightarrow (1+\lambda)(1-\lambda)(-5-\lambda)+0-(1+\lambda)\cdot0-(1-\lambda)\cdot4+(5+\lambda)\cdot0=0$

$\Rightarrow (\lambda-1)(\lambda+3)^2=0 \Rightarrow \lambda=1,-3$

$\therefore$ Two pair of straight lines can be drawn.

61. (A) → (s), (B) → (r), (C) → (q), (D) → (p)

(A) The chord with mid point (h, k)

$\dfrac{hx}{a^2}+\dfrac{ky}{b^2}=\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}$ $\therefore$ Locus is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{y}{b}$

(B) (h, k) be the end of the latus rectum.

$h = ae,\ k = a(1-e^2)$

$h^2 = -a(k-a) \Rightarrow x^2 = -a(y-a)$, parabola.

(C) $y-mx=\pm\sqrt{a^2m^2+b^2}$ $\therefore$ $x^2+y^2=a^2$

(D) Line $\dfrac{x}{a}+\dfrac{y}{b}=1$ passes through fixed point (α, β)

$\therefore$ $\dfrac{\alpha}{a}+\dfrac{\beta}{b}=1$; $h=\dfrac{a}{2}, k=\dfrac{b}{2}$

$\dfrac{\alpha}{x}+\dfrac{\beta}{y}=2\Rightarrow\left(x-\dfrac{\alpha}{2}\right)\left(y-\dfrac{\beta}{2}\right)=\dfrac{\alpha\beta}{4}$

62. (5) : Equation can be rewritten as

$\sqrt{(x-2)^2+(y+3)^2}=\dfrac{13}{5}\left|\dfrac{12x-5y+1}{13}\right|$ So, $e=\dfrac{13}{5}$.

63. (1) : The given curve represents the point $\left(-\frac{1}{2}, 2\right)$
$\therefore$ Minimum distance = 1.

64. (8) : $(x-2)^2+y^2=\left(\dfrac{1}{2}\right)^2\dfrac{(x+\sqrt{3}y+10)^2}{4}\Rightarrow e = 1/2$

Perpendicular distance from (2, 0) to $x+\sqrt{3}y+10=0$ is $\dfrac{a}{e}-ae \Rightarrow 2a-\dfrac{a}{2}=6 \Rightarrow a=4$

Distance between directrices $=\dfrac{2a}{e}=16=K$

65. (4) : Director circle of $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ is $x^2+y^2=25$

The director circle will cut the ellipse $\dfrac{x^2}{50}+\dfrac{y^2}{20}=1$ at 4 points.

66. (7) : The equation of the tangent at $(5\cos\theta, 2\sin\theta)$ is $\dfrac{x}{5}\cos\theta+\dfrac{y}{2}\sin\theta=1$

If it is a tangent to the circle then $\dfrac{1}{\sqrt{\dfrac{\cos^2\theta}{25}+\dfrac{\sin^2\theta}{4}}}=4$

$\Rightarrow \cos\theta=\dfrac{10}{4\sqrt{7}}, \sin\theta=\dfrac{\sqrt{3}}{2\sqrt{7}}$

Let A and B be the points where the tangent meets the coordinate axis then

$A\left(\dfrac{5}{\cos\theta}, 0\right)$, $B\left(0, \dfrac{2}{\sin\theta}\right)$; $L=\sqrt{\dfrac{25}{\cos^2\theta}+\dfrac{4}{\sin^2\theta}}=\dfrac{14}{\sqrt{3}}$

Monthly Test Drive-2 CLASS XI ANSWER KEY

1. (c)	**2.** (a)	**3.** (d)	**4.** (a)	**5.** (b)
6. (c)	**7.** (b)	**8.** (a,c)	**9.** (b,c)	**10.** (d)
11. (b,c)	**12.** (a,c,d)	**13.** (a)	**14.** (a)	**15.** (b)
16. (d)	**17.** (4)	**18.** (5)	**19.** (1)	**20.** (1)

CBSE warm-up!

CLASS-XI

Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2020-21.

Series 2

Trigonometric Functions and Principle of Mathematical Induction

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. Find the radian measure of 625°.
(a) 11.82 radian (b) 10.91 radian
(c) 9.91 radian (d) None of these

2. If $\frac{\sin x}{\cos x} \times \frac{\sec x}{\operatorname{cosec} x} \times \frac{\tan x}{\cot x} = 9$, where $x \in \left(0, \frac{\pi}{2}\right)$, then the value of x is equal to
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

3. Find the value of
$\sin(40° + \theta)\cos(10° + \theta) - \cos(40° + \theta)\sin(10° + \theta)$.
(a) 1/2 (b) 1/4 (c) 1 (d) 0

4. The value of x in $(0, \pi/2)$ satisfying the equation $\sin x \cos x = \frac{1}{4}$ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{12}$

5. In any ΔABC, if $a = 18$, $b = 24$ and $c = 30$, then find $\cos B$.
(a) $\frac{3}{5}$ (b) $\frac{2}{5}$ (c) $\frac{1}{5}$ (d) $\frac{1}{4}$

6. For natural number n, $2^n (n-1)! < n^n$, if
(a) $n < 2$ (b) $n > 2$ (c) $n \geq 2$ (d) $n > 3$

7. If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of $\theta + \phi$ is
(a) π (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) 0

8. Find the remainder, when 5^{4n} is divided by 13.
(a) 1 (b) 2 (c) 3 (d) 4

9. If $\sin\theta + \cos\theta = 1$, then the value of $\sin 2\theta$ is equal to
(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) -1

10. According to principle of mathematical induction, if $P(k+1) = m^{(k+1)} + 5$ is true, then

(a) $P(k) = m^k$

(b) $P(k) = m^k + 5$

(c) $P(k) = m^{k+2} + 5$

(d) None of these

(Q. 11-Q.15) Fill in the blanks.

11. If $P(n) : 2n < n!$, $n \in N$, then $P(n)$ is true for all $n \geq$ _____.

12. If in a circle of diameter 40 cm the length of a chord is 20 cm, then the length of minor arc corresponding to the chord is _____.

OR

If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then the value of $4\tan^2\theta - 3\,\text{cosec}^2\theta$ is _____.

13. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is _____.

14. The value of $\cos 55° + \cos 125° + \cos 300°$ is _____.

15. If $\sin\theta = \sin 15° + \sin 45°$, where $0° < \theta < 180°$, then θ is equal to is _____.

(Q. 16-Q.20) Answer the following questions.

16. Convert $40°\,20'$ into radian measure.

17. Express the following as a sum or difference:

$2\sin\frac{5\theta}{2}\cos\frac{7\theta}{2}$

18. Solve : $7\cos^2\theta + 3\sin^2\theta = 4$

OR

Write the general solution of $\cos x = \frac{1}{2}$.

19. In ΔABC, if $\angle A = 45°$, $\angle B = 60°$ and $\angle C = 75°$, find the ratio of its sides.

20. Let $P(n)$ be the statement "$3^n > n$". If $P(n)$ is true, prove that $P(n+1)$ is true.

SECTION - B

21. Find the angle in radians through which a pendulum swings if its length is 75 cm and the tip describes an arc of length 21 cm.

22. Find the values of $\cos\theta$ and $\tan\theta$, if

$\sin\theta = -\frac{3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$.

OR

If $A + B + C = \pi$, prove that

$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

23. If $\sin(A+B) = \frac{\sqrt{3}}{2}$, $\cos(A-B) = \frac{\sqrt{3}}{2}$, then find A and B if they lie in the first quadrant.

24. Prove that :

$\sqrt{2+\sqrt{2+2\cos 4x}} = 2\cos x, 0 < x < \frac{\pi}{4}$

OR

Solve the following equation :

$\tan 3\theta + \tan\theta = 2\tan 2\theta$

25. Using induction, prove that for all $n \in N$.

$5^n - 5$ is divisible by 4.

26. $5^{2n} - 1$ is divisible by 24 where n is a positive integer.

SECTION - C

27. Solve: $\sec x - \tan x = \sqrt{3}$.

28. If $\cos(\alpha+\beta) = \frac{4}{5}$, $\sin(\alpha-\beta) = \frac{5}{13}$ and α, β lie between 0 and $\frac{\pi}{4}$, prove that $\tan 2\alpha = \frac{56}{33}$

OR

Solve the trigonometric equation :

$4\sin x \sin 2x \sin 4x = \sin 3x$

29. Prove by the method of induction that :

$2 + 5 + 8 + 11 + \ldots\ldots + (3n-1) = \frac{1}{2}n(3n+1), n \in N.$

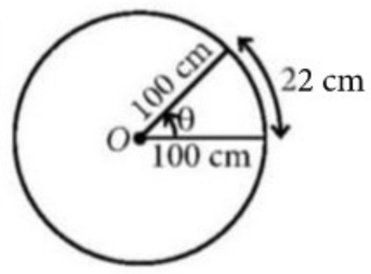
30. Using the principle of mathematical induction, prove that $(n^2 + n)$ is even for all $n \in N$.

31. Find the value of $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$.

OR

Evaluate : $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$

32. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm as shown in figure. $\left(\text{Use } \pi = \frac{22}{7}\right)$.



SECTION - D

33. Prove that:

$$\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$$

34. Prove that : $\cos x+\cos y+\cos z+\cos(x+y+z)$

$$=4\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{y+z}{2}\right)\cos\left(\frac{z+x}{2}\right)$$

OR

Solve the following equation :

$\sin 2x+\sin 4x=\cos x+\cos 3x$

35. For all positive integers n, prove that $\frac{n^7}{7}+\frac{n^5}{5}+\frac{2n^3}{3}-\frac{n}{105}$ is an integer.

OR

Prove by using principle of induction that

$\cos\alpha+\cos 2\alpha+\ldots\ldots+\cos n\alpha$

$$=\sin\frac{n\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}\cos\frac{(n+1)\alpha}{2}$$

36. Using principle of mathematical induction, prove that:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\ldots\left(1-\frac{1}{n^2}\right)$$

$=\frac{n+1}{2n},\forall n\geq 2$ where n is a natural number.

SOLUTIONS

1. (b) : $625^\circ=\left(625\times\frac{\pi}{180}\right)^c=10.91$ radian

$\left(\because 1^\circ=\frac{\pi}{180}\text{ radian}\right)$

2. (b) : $\frac{\sin x}{\cos x}\times\frac{\sec x}{\operatorname{cosec}x}\times\frac{\tan x}{\cot x}=9\Rightarrow\tan^4x=9$

$\Rightarrow\tan^2x=3\Rightarrow\tan x=\pm\sqrt{3}\Rightarrow x=\frac{\pi}{3}\in\left(0,\frac{\pi}{2}\right)$

3. (a) : We have,

$\sin(40^\circ+\theta)\cdot\cos(10^\circ+\theta)-\cos(40^\circ+\theta)\cdot\sin(10^\circ+\theta)$

$=\sin\{(40^\circ+\theta)-(10^\circ+\theta)\}=\sin30^\circ=\frac{1}{2}$

4. (d) : $\sin x\cos x=\frac{1}{4};\ x\in\left(0,\frac{\pi}{2}\right)$

$\Rightarrow\sin 2x=\frac{1}{2}\Rightarrow 2x=\frac{\pi}{6}\qquad\left[\because\ x\in\left(0,\frac{\pi}{2}\right)\right]$

$\Rightarrow x=\frac{\pi}{12}$

5. (a) : Given, $a=18$, $b=24$ and $c=30$

$\Rightarrow\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{324+900-576}{1080}=\frac{3}{5}$

6. (b) : The condition $2^n(n-1)!<n^n$ is satisfied for $n>2$.

7. (c) : We have, $\tan\theta=\frac{1}{2}$, $\tan\phi=\frac{1}{3}$

Now, $\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta.\tan\phi}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=\frac{\frac{5}{6}}{\frac{5}{6}}=1$

$\therefore\ \tan(\theta+\phi)=\tan\frac{\pi}{4}\Rightarrow\ \theta+\phi=\frac{\pi}{4}$

8. (a) : For $n=1$,

$5^4=625=(624+1)=(48\times13)+1$

i.e, 5^4 leaves 1 as remainder, when divided by 13.

9. (b) : We have, $\sin\theta+\cos\theta=1$

$\Rightarrow(\sin\theta+\cos\theta)^2=1$

$\Rightarrow\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=1$

$\Rightarrow 1+\sin 2\theta=1\Rightarrow\sin 2\theta=0$

10. (b) : By the principle of mathematical induction, if a statement is true for any number $n=k$, then for its successor $k+1$, statement also satisfies, provided the statement is true for $n=1$. So, the required answer is $P(k)=m^k+5$.

11. We have $P(n):2n<n!$

For $n=1$, $2\not<1$ $\qquad[\because\ 1!=1]$

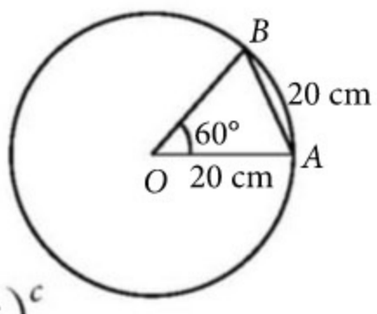
For $n=2$, $4\not<2$ $\qquad[\because\ 2!=2\times1=2]$

For $n=3$, $6\not<6$ $\qquad[\because\ 3!=3\times2\times1=6]$

For $n=4$, $8<24$ $\qquad[\because\ 4!=4\times3\times2\times1=24]$

So, $P(n):2n<n!$ is true for all $n\geq4$.

12. Let arc $AB=s$. It is given that $OA=20$ cm and chord $AB=20$ cm. Therefore, ΔOAB is an equilateral triangle. Hence,



$$\angle AOB=60^\circ=\left(60\times\frac{\pi}{180}\right)^c=\left(\frac{\pi}{3}\right)^c$$

$\therefore\quad\theta=\frac{\text{arc}}{\text{radius}}\Rightarrow\frac{\pi}{3}=\frac{s}{20}\Rightarrow s=\frac{20\pi}{3}$ cm.

OR

Since θ lies in the third quadrant. Therefore, $\sin\theta$ is negative and $\tan\theta$ is positive.

Now, $\sin\theta=\pm\sqrt{1-\cos^2\theta}$

$\Rightarrow\ \sin\theta=-\sqrt{1-\frac{1}{4}}=-\frac{\sqrt{3}}{2}\Rightarrow\operatorname{cosec}\theta=\frac{-2}{\sqrt{3}}$

And, $\tan\theta=\frac{\sin\theta}{\cos\theta}=\sqrt{3}$.

Hence, $4\tan^2\theta-3\operatorname{cosec}^2\theta=4\times3-3\times\frac{4}{3}=8$.

13. Let $P(n) : x^2 - 1$ is divisible by $x - k$.

$P(1) : x - 1$ is divisible by $x - k$.

Since $x - 1$ is divisible by $x - 1$ the least integral value of k is 1.

14. $\cos 55^\circ + \cos 125^\circ + \cos 300^\circ$

$= 2\cos 90^\circ \cos 35^\circ + \cos(360^\circ - 60^\circ) = 0 + \cos 60^\circ = \frac{1}{2}$

OR

Let the three angles be $3x$, $4x$ and $5x$ degrees, then

$3x + 4x + 5x = 180^\circ$

$\Rightarrow \quad 12x = 180^\circ \Rightarrow x = 15^\circ$

Greatest angle $= 5x = 75^\circ$

$$= 75 \times \frac{\pi}{180} \text{radians} = \frac{5\pi}{12} \text{ radians}$$

15. We have, $\sin\theta = \sin 15^\circ + \sin 45^\circ$

$$\sin\theta = 2\sin\left(\frac{15+45}{2}\right)^\circ \cos\left(\frac{15-45}{2}\right)^\circ$$

$$= 2\sin 30^\circ \cos 15^\circ = 2 \times \frac{1}{2}\cos 15^\circ = \cos 15^\circ$$

$$\Rightarrow \quad \sin\theta = \sin(90-15)^\circ \quad \Rightarrow \quad \theta = 75^\circ$$

16. $40^\circ 20' = \left(40\frac{20}{60}\right)^\circ = \left(\frac{121}{3}\right)^\circ$

Now, $1^\circ = \left(\frac{\pi}{180}\right)^c \Rightarrow \left(\frac{121}{3}\right)^\circ = \left(\frac{121}{3} \times \frac{\pi}{180}\right)^c = \left(\frac{121\pi}{540}\right)^c$

17. $2\sin\frac{5\theta}{2}\cos\frac{7\theta}{2} = \sin\left(\frac{5\theta}{2} + \frac{7\theta}{2}\right) + \sin\left(\frac{5\theta}{2} - \frac{7\theta}{2}\right)$

$[\because 2\sin A \cos B = \sin(A + B) + \sin(A - B)]$

$= \sin 6\theta + \sin(-\theta) = \sin 6\theta - \sin\theta \quad [\because \sin(-q) = -\sin q]$

18. We have, $7\cos^2\theta + 3\sin^2\theta = 4$

$\Rightarrow \quad 7(1 - \sin^2\theta) + 3\sin^2\theta = 4 \Rightarrow 4\sin^2\theta = 3$

$$\Rightarrow \quad \sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in Z$$

OR

We have, $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$$\therefore \quad x = 2n\pi \pm \frac{\pi}{3} \text{ where } n \in Z$$

19. By sine formula, we have

$a : b : c = \sin A : \sin B : \sin C$

$= \sin 45^\circ : \sin 60^\circ : \sin 75^\circ$

$$= \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} : \frac{\sqrt{6}}{2\sqrt{2}} : \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\Rightarrow \quad a : b : c = 2 : \sqrt{6} : \sqrt{3}+1$$

20. It is given that $P(n)$ is true *i.e.*, $3^n > n$.

We have to prove that $P(n + 1)$ is true *i.e.*, $3^{n+1} > (n + 1)$

Now, $3^n > n$

$\Rightarrow \quad 3 \cdot 3^n > 3n$ (Multiplying by 3 on both sides)

$\Rightarrow \quad 3^{n+1} > n + 2n$

$\Rightarrow \quad 3^{n+1} > n + 1$ $[\because 2n > 1 \text{ for all } n \in N]$

which shows that $P(n + 1)$ is true.

21. Radius of arc $(r) = 75$ cm

Length of arc $(l) = 21$ cm

We know that $l = r\theta \Rightarrow \theta = \frac{l}{r}$

$\therefore \quad$ Angle, $\theta = \frac{21}{75} = 0.28$ radians

22. We have, $\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1 - \sin^2\theta}$

In the third quadrant $\cos\theta$ is negative and $\tan\theta$ is positive.

$$\therefore \quad \cos\theta = -\sqrt{1 - \sin^2\theta} = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5}$$

$$\text{and, } \tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{5} \times -\frac{5}{4} = \frac{3}{4}$$

OR

We have, L.H.S $= \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A + B)\cos(A - B) + 2\cos^2 C - 1$

$= 2\cos(\pi - C)\cos(A - B) + 2\cos^2 C - 1$

$= -2\cos C \cos(A - B) + 2\cos^2 C - 1$

$= -2\cos C[\cos(A - B) - \cos C] - 1$

$= -1 - 2\cos C[\cos(A - B) - \cos\{\pi - (A + B)\}]$

$= -1 - 2\cos C[\cos(A - B) + \cos(A + B)]$

$= -1 - 2\cos C[2\cos A \cos B]$

$= -1 - 4\cos A \cos B \cos C$ = R.H.S.

23. $\sin(A + B) = \frac{\sqrt{3}}{2} = \sin 60^\circ$

$\therefore \quad A + B = 60^\circ$ ...(i)

$\text{and } \cos(A - B) = \frac{\sqrt{3}}{2} = \cos 30^\circ$

$\therefore \quad A - B = 30^\circ$ ...(ii)

Solving (i) and (ii), we get $A = 45^\circ$ and $B = 15^\circ$.

24. L.H.S. $= \sqrt{2 + \sqrt{2 + 2\cos 4x}}$

$$= \sqrt{2 + \sqrt{2(1 + \cos 4x)}} = \sqrt{2 + \sqrt{2 \cdot 2\cos^2 2x}}$$

$$= \sqrt{2 + 2\cos 2x} = \sqrt{2(1 + \cos 2x)}$$

$= \sqrt{2 \cdot 2\cos^2 x} = 2\cos x$ = R.H.S.

OR

$\tan 3\theta - \tan 2\theta = \tan 2\theta - \tan \theta$

$\Rightarrow \quad 2\sin^2\theta \sin 2\theta = 0$

$\Rightarrow \quad \sin\theta = 0$ or $\sin 2\theta = 0$

$\Rightarrow \quad \theta = n\pi$ or $2\theta = m\pi$, where $n, m \in Z$

$\Rightarrow \quad \theta = n\pi$ or $\theta = \frac{m\pi}{2}$, where $m, n \in Z$

25. Let $P(n) : 5^n - 5$ is divisible by 4

First we prove that the statement is true for $n = 1$,

$P(1) : 5^1 - 5 = 0$, which is divisible by 4.

$\therefore \quad P(1)$ is true

Assume $P(k)$ is true for some positive integer k

$\therefore \quad 5^k - 5$ is divisible by 4

$\Rightarrow \quad 5^k - 5 = 4d, d \in N$ ...(i)

Now, we shall prove $P(k + 1)$ is true

For this we have to prove $5^{k+1} - 5$ is divisible by 4.

Consider, $5^{k+1} - 5 = 5^k \cdot 5 - 5$

$= (4d + 5)\cdot 5 - 5$ (From (i))

$= 20d + 25 - 5$

$= 20d + 20 = 4(5d + 5)$, which is divisible by 4

Hence, $P(k + 1)$ is true, whenever $P(k)$ is true

Hence, by principle of mathematical induction $P(n)$ is true $\forall n \in N$.

26. Let $P(n) = 5^{2n} - 1$ is divisible by 24, $n \in N$.

First we will prove that $P(n)$ is true for $n = 1$.

$P(1) = 5^2 - 1 = 24$ is divisible by 24, which is true.

Let $P(k)$ be true for some positive integer k.

Then, $P(k) = 5^{2k} - 1$ is divisible by 24

i.e., $5^{2k} - 1 = 24\, l$, where $l \in N$ ...(i)

Now, we have to prove that $P(k + 1)$ is true

i.e., $P(k + 1) = 5^{2(k+1)} - 1$ is divisible by 24.

$P(k + 1) = 5^{2(k+1)} - 1$

$= 5^{2k} \cdot 5^2 - 1 = (24\, l + 1) \cdot 5^2 - 1$ [From (i)]

$= 25 \cdot 24\, l + 25 - 1$

$= 24(25\, l + 1)$, which is divisible by 24.

Thus, $P(k)$ is true $\Rightarrow P(k + 1)$ is true

Hence, by principle of mathematical induction $P(n)$ is true, $\forall n \in N$.

27. We have, $\sec x - \tan x = \sqrt{3} \Rightarrow \quad \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \sqrt{3}$

$\Rightarrow (1 - \sin x) = \sqrt{3}\cos x \Rightarrow \sqrt{3}\cos x + \sin x = 1$ ...(i)

Dividing both sides of (i) by 2, we get

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad \left(x - \frac{\pi}{6}\right) = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in Z$$

$[\because \cos\theta = \cos\alpha \quad \Rightarrow \theta = 2n\pi \pm \alpha]$

$$\Rightarrow \quad \left(x - \frac{\pi}{6}\right) = \left(2n\pi + \frac{\pi}{3}\right) \text{ or } \left(x - \frac{\pi}{6}\right) = \left(2n\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow \quad x = 2n\pi + \frac{\pi}{3} + \frac{\pi}{6} \text{ or } x = 2n\pi - \frac{\pi}{3} + \frac{\pi}{6}, \text{where } n \in Z$$

$$\Rightarrow \quad x = \left(2n\pi + \frac{\pi}{2}\right) \text{ or } x = \left(2n\pi - \frac{\pi}{6}\right), \text{ where } n \in Z$$

$$\Rightarrow \quad x = \left(2n\pi - \frac{\pi}{6}\right), \text{ where } n \in Z$$

$$\left[\because \sec x \text{ is not defined when } x = \left(2n\pi + \frac{\pi}{2}\right)\right].$$

Hence, the general solution is $x = \left(2n\pi - \frac{\pi}{6}\right)$, where $n \in Z$.

28. It is given that α, β lie between 0 and $\pi/4$. Therefore, $-\pi/4 < \alpha - \beta < \pi/4$ and $0 < \alpha + \beta < \pi/2$.

So, $\cos(\alpha - \beta)$ and $\sin(\alpha + \beta)$ are positive.

Now, $\sin(\alpha + \beta) = \sqrt{1 - \cos^2(\alpha + \beta)}$

$$\Rightarrow \quad \sin(\alpha + \beta) = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

and, $\cos(\alpha - \beta) = \sqrt{1 - \sin^2(\alpha - \beta)}$

$$\Rightarrow \quad \cos(\alpha - \beta) = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$$

$$\therefore \quad \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{3/5}{4/5} = \frac{3}{4} \text{ and,}$$

$$\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{5}{12}$$

Now, $\tan 2\alpha = \tan\{(\alpha + \beta) + (\alpha - \beta)\}$

$$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} = \frac{56}{33}$$

OR

We have, $4 \sin x \sin 2x \sin 4x = \sin 3x$

$\Rightarrow \quad 2(2 \sin x \sin 2x)\sin 4x = \sin 3x$

$\Rightarrow \quad 2[\cos(x - 2x) - \cos(x + 2x)]\sin 4x = \sin 3x$

$\Rightarrow \quad 2(\cos x - \cos 3x)\sin 4x = \sin 3x$

$\Rightarrow \quad 2\cos x \sin 4x - 2\cos 3x \sin 4x = \sin 3x$

$\Rightarrow$ $[\sin(x+4x)-\sin(x-4x)]$
$-[\sin(3x+4x)-\sin(3x-4x)]=\sin 3x$

$\Rightarrow$ $\sin 5x+\sin 3x-\sin 7x-\sin x=\sin 3x$

$\Rightarrow$ $\sin 5x-\sin 7x=\sin x$

$\Rightarrow$ $-2\cos 6x\sin x=\sin x$

$\Rightarrow$ $\sin x(1+2\cos 6x)=0$

$\Rightarrow$ $\sin x=0;\ 2\cos 6x=-1 \quad\Rightarrow\quad \cos 6x=-\frac{1}{2}$

$\Rightarrow$ $x=n\pi;\ \cos 6x=\cos\left(\pi-\frac{\pi}{3}\right)$

$\Rightarrow$ $x=n\pi;\ 6x=2m\pi\pm\left(\frac{3\pi-\pi}{3}\right)$

$\Rightarrow$ $x=n\pi;\ x=\frac{1}{6}\left(2m\pi\pm\frac{2\pi}{3}\right)=\frac{m\pi}{3}\pm\frac{\pi}{9}$, where m, $n\in Z$.

29. Let $P(n): 2+5+8+11+.....+(3n-1)=\frac{1}{2}n(3n+1),\ n\in N$

First we prove that $P(n)$ is true for $n=1$

$P(1): 3(1)-1=\frac{1}{2}(1)(3+1)$

$\Rightarrow$ $2=2$, which is true.

Let $P(k)$ be true, for some positive integer k.

i.e., $P(k): 2+5+8+11+.....+(3k-1)=\frac{1}{2}k(3k+1)$ ...(i)

Now, we have to show that $P(k+1)$ is true.

i.e., $P(k+1)=2+5+8+11+.....+(3k-1)+[3(k+1)-1]$

$=\frac{1}{2}(k+1)[3(k+1)+1]$

L.H.S. $=\frac{1}{2}k(3k+1)+3k+2$ (Using (i))

$=\frac{3k^2+k+6k+4}{2}=\frac{3k^2+7k+4}{2}$

$=\frac{3k^2+4k+3k+4}{2}=\frac{k(3k+4)+1(3k+4)}{2}$

$=\frac{(k+1)(3k+4)}{2}=\frac{1}{2}(k+1)[3(k+1)+1]=$ R.H.S.

Hence, $P(k)$ is true $\Rightarrow P(k+1)$ is true.

$\therefore$ By principle of mathematical induction $P(n)$ is true $\forall n\in N$.

30. Let $P(n): (n^2+n)$ is even.

For $n=1$, the given expression becomes $(1^2+1)=2$, which is even.

So, the given statement is true for $n=1$,

i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: (k^2+k) is even

$\Rightarrow$ $(k^2+k)=2m$ for some natural number m. ...(i)

Now, $(k+1)^2+(k+1)=k^2+3k+2$

$=(k^2+k)+2(k+1)$

$=2m+2(k+1)$ (Using (i))

$=2[m+(k+1)]$, which is clearly even.

$\therefore$ $P(k+1)$: $(k+1)^2+(k+1)$ is even

$\Rightarrow$ $P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n\in N$.

31. We have, $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$

$=3(1-\sin 2x)^2+6(1+\sin 2x)+4[(\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x)]$

$=3(1-2\sin 2x+\sin^2 2x)+6+6\sin 2x+4\left[1-\frac{3}{4}\sin^2 2x\right]$

$=13+3\sin^2 2x-3\sin^2 2x=13$

OR

Using $\cos\alpha+\cos 2\alpha+\cos 3\alpha+\cos 4\alpha...+\cos n\alpha$

$=\frac{\cos\left(\frac{n+1}{2}\alpha\right)\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$, we get

$\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{7\pi}{7}$

$=\frac{\cos\left(\frac{7+1}{2}\frac{\pi}{7}\right)\sin\left(\frac{7\pi}{2\cdot 7}\right)}{\sin\left(\frac{\pi}{2\cdot 7}\right)}$ $\left[\because\ \alpha=\frac{\pi}{7}\text{ and }n=7\right]$

$=\frac{\cos\frac{4\pi}{7}\sin\frac{\pi}{2}}{\sin\frac{\pi}{14}}=\frac{\sin\left(\frac{\pi}{2}-\frac{4\pi}{7}\right)}{\sin\frac{\pi}{14}}$

$=\frac{\sin\left(-\frac{\pi}{14}\right)}{\sin\frac{\pi}{14}}=\frac{-\sin\frac{\pi}{14}}{\sin\frac{\pi}{14}}=-1$

32. We know that $l=r\theta$

where, l = Length of arc = 22 cm

r = Radius of circle = 100 cm

θ = Angle subtended at the centre

$\therefore$ $\theta=\frac{l}{r}=\frac{22}{100}=0.22$ radian

$$= \left(0.22\times\frac{180}{\pi}\right)^\circ = \left(\frac{0.22\times180\times7}{22}\right)^\circ$$

$$= \left(\frac{22}{100}\times\frac{180\times7}{22}\right)^\circ = \left(\frac{126}{10}\right)^\circ = 12^\circ36'$$

33. We observed that

$$\sin^4\frac{7\pi}{8} = \sin^4\left(\pi-\frac{\pi}{8}\right) = \sin^4\frac{\pi}{8} \text{ and,}$$

$$\sin^4\frac{5\pi}{8} = \sin^4\left(\pi-\frac{3\pi}{8}\right) = \sin^4\frac{3\pi}{8}$$

$$\therefore \text{ L.H.S.} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$$

$$= \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{\pi}{8}$$

$$= 2\left\{\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}\right\} = 2\left\{\left(\sin^2\frac{\pi}{8}\right)^2 + \left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$

$$= 2\left[\left\{\frac{1-\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1-\cos\frac{3\pi}{4}}{2}\right\}^2\right] \left[\because \frac{1-\cos2\theta}{2} = \sin^2\theta\right]$$

$$= \frac{2}{4}\left\{\left(1-\cos\frac{\pi}{4}\right)^2 + \left(1-\cos\frac{3\pi}{4}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{2}-\sqrt{2}\right)+\left(1+\frac{1}{2}+\sqrt{2}\right)\right\} = \frac{3}{2} = \text{R.H.S.}$$

34. L.H.S. $= \cos x + \cos y + \cos z + \cos(x+y+z)$

$= (\cos x + \cos y) + (\cos z + \cos(x+y+z))$

$$= 2\cos\frac{(x+y)}{2}\cos\frac{(x-y)}{2} + 2\cos\frac{(z+x+y+z)}{2}\cos\frac{(z-x-y-z)}{2}$$

$$= 2\cos\frac{(x+y)}{2}\cos\frac{(x-y)}{2} + 2\cos\frac{(x+y+2z)}{2}\cos\frac{(x+y)}{2}$$

$$= 2\cos\frac{(x+y)}{2}\left[\cos\frac{(x-y)}{2} + \cos\frac{(x+y+2z)}{2}\right]$$

$$= 2\cos\frac{(x+y)}{2}\left[2\cos\frac{\left(\frac{x-y}{2}+\frac{x+y+2z}{2}\right)}{2}\cos\frac{\left(\frac{x+y+2z}{2}-\frac{x-y}{2}\right)}{2}\right]$$

$$= 2\cos\frac{(x+y)}{2}\left(2\cos\frac{(x+z)}{2}\cos\frac{(y+z)}{2}\right)$$

$$= 4\cos\frac{(x+y)}{2}\cos\frac{(x+z)}{2}\cos\frac{(y+z)}{2} = \text{R.H.S.}$$

OR

$\sin 2x + \sin 4x = \cos x + \cos 3x$

$\Rightarrow$ $2\sin 3x\cos(-x) = 2\cos 2x\cos(-x)$

$\Rightarrow$ $\sin 3x\cos x = \cos 2x\cos x$

$\Rightarrow$ $\cos x(\sin 3x - \cos 2x) = 0$

$\Rightarrow$ Either $\cos x = 0$ or $\sin 3x = \cos 2x$

When $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$

When $\sin 3x = \cos 2x$

$$\Rightarrow \sin 3x = \sin\left(\frac{\pi}{2}-2x\right)$$

$$\Rightarrow 3x = n\pi + (-1)^n\left(\frac{\pi}{2}-2x\right)$$

$$\Rightarrow 3x = n\pi + (-1)^n\frac{\pi}{2} - 2(-1)^n x$$

$$\Rightarrow [3+2(-1)^n]x = n\pi + (-1)^n\frac{\pi}{2}$$

$$\therefore \quad x = \frac{n\pi + (-1)^n\cdot\frac{\pi}{2}}{3+2(-1)^n}, n \in Z$$

35. Let $P(n) = \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer.

First we prove that the statement is true for $n = 1$

$$\therefore P(1) = \frac{(1)^7}{7} + \frac{(1)^5}{5} + \frac{2(1)^3}{3} - \frac{(1)}{105} = \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105}$$

$$= \frac{15+21+70-1}{105} = \frac{106-1}{105} = \frac{105}{105} = 1$$

$\therefore$ $P(1)$ is true

Let $P(n)$ be true for $n = k$

$$\therefore \quad P(k) = \frac{k^7}{7} + \frac{k^5}{5} + \frac{2k^3}{3} - \frac{k}{105} \text{ is an integer.}$$

$$\therefore \quad \frac{k^7}{7} + \frac{k^5}{5} + \frac{2k^3}{3} - \frac{k}{105} = m, m \in Z \qquad \text{...(i)}$$

Now we shall prove that $P(k+1)$ is true.

$$\therefore P(k+1) = \frac{(k+1)^7}{7} + \frac{(k+1)^5}{5} + \frac{2(k+1)^3}{3} - \frac{(k+1)}{105}$$

$$= \frac{1}{7}(k^7+7k^6+21k^5+35k^4+35k^3+21k^2+7k+1) + \frac{1}{5}(k^5+5k^4+10k^3+10k^2+5k+1) + \frac{2}{3}(k^3+3k^2+3k+1) - \frac{k}{105} - \frac{1}{105}$$

$$= \left(\frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}\right)+k^6+3k^5+6k^4+7k^3+7k^2+4k+1$$

$$\left[\because \frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105}=1\right]$$

$= m + k^6 + 3k^5 + 6k^4 + 7k^3 + 7k^2 + 4k + 1$ [Using (i)]

which is an integer

Thus, $P(k)$ is true $\Rightarrow P(k+1)$ is true

Hence, by principle of mathematical induction, $P(n)$ is true for all positive integer n.

OR

Let $P(n)$: $\cos\alpha + \cos 2\alpha + \ldots\ldots + \cos n\alpha$

$$= \sin\frac{n\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}\cos\frac{(n+1)\alpha}{2}$$

When $n = 1$, L.H.S. $= \cos\alpha$

and R.H.S. $= \sin\frac{\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}\cos\alpha = \cos\alpha$

$\therefore$ L.H.S. = R.H.S.

Hence $P(1)$ is true.

Let $P(m)$ be true.

$\Rightarrow \cos\alpha + \cos 2\alpha + \ldots\ldots + \cos m\alpha$

$$= \sin\frac{m\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}\cos\frac{(m+1)\alpha}{2} \quad \ldots(i)$$

To prove $P(m+1)$ is true *i.e.*,

$\cos\alpha + \cos 2\alpha + \ldots\ldots + \cos m\alpha + \cos(m+1)\alpha$

$$= \sin\frac{(m+1)\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}\cos\frac{(m+2)\alpha}{2}$$

Adding $\cos(m+1)\alpha$ to both sides of (i), we get

$\cos\alpha + \cos 2\alpha + \ldots\ldots + \cos m\alpha + \cos(m+1)\alpha$

$$= \sin\frac{m\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}\cos\frac{(m+1)\alpha}{2}+\cos(m+1)\alpha$$

$$= \operatorname{cosec}\frac{\alpha}{2}\left[\sin\frac{m\alpha}{2}\cos\frac{(m+1)\alpha}{2}+\cos(m+1)\alpha\sin\frac{\alpha}{2}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\left[2\cos\frac{(m+1)\alpha}{2}\sin\frac{m\alpha}{2} + 2\cos(m+1)\alpha\sin\frac{\alpha}{2}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\left[\sin\frac{(2m+1)\alpha}{2}-\sin\frac{\alpha}{2} + \sin\left(m+1+\frac{1}{2}\right)\alpha-\sin\left(m+1-\frac{1}{2}\right)\alpha\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\left[\sin\frac{(2m+1)\alpha}{2}-\sin\frac{\alpha}{2} + \sin\left(\frac{2m+3}{2}\right)\alpha-\sin\left(\frac{2m+1}{2}\right)\alpha\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\left[\sin\left(\frac{2m+3}{2}\right)\alpha-\sin\frac{\alpha}{2}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}2\cos\left(\frac{2m+3+1}{4}\right)\alpha\cdot\sin\left(\frac{2m+2}{4}\right)\alpha$$

$$= \operatorname{cosec}\frac{\alpha}{2}\cos\left(\frac{m+2}{2}\right)\alpha\sin\left(\frac{m+1}{2}\right)\alpha$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m+1)$ is true whenever $P(m)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true for all $n \in N$.

36. Let $P(n)$: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)$

$$= \frac{n+1}{2n}, \forall n \geq 2$$

First we prove $P(n)$ is true for $n = 2$

$$P(1): \left(1-\frac{1}{2^2}\right) = \frac{2+1}{2\times 2} \Rightarrow \frac{3}{4} = \frac{3}{4}, \text{ which is true.}$$

Assume $P(k)$ is true, for some positive integer k.

$$i.e., \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\ldots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k} \quad \ldots(i)$$

Now, we shall prove that $P(k+1)$ is also true.

For this we have to prove

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$$

$$\text{L.H.S.} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right) \quad \text{(From (i))}$$

$$= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\left(\frac{k^2+1+2k-1}{(k+1)^2}\right) = \left(\frac{k+1}{2k}\right)\left(\frac{k^2+2k}{(k+1)^2}\right)$$

$$= \frac{k(k+2)}{2k(k+1)} = \frac{(k+2)}{2(k+1)} = \text{R.H.S.}$$

Hence, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true $\forall\, n \in N$.

MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Series 2 : Complex Numbers and Quadratic Equations

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. $\left(\dfrac{1+i\sin\frac{\pi}{8}+\cos\frac{\pi}{8}}{1-i\sin\frac{\pi}{8}+\cos\frac{\pi}{8}}\right)^8$ equals

(a) 2^8 (b) 0

(c) -1 (d) 1

2. If a, b, c are in G.P., then the equations $ax^2 + 2bx + c = 0$, $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

3. For positive integers n_1 and n_2 the value of the expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$ where $i = \sqrt{-1}$ is a real number iff

(a) $n_1 = n_2$ (b) $n_2 = n_2 - 1$

(c) $n_1 = n_2 + 1$ (d) $\forall\ n_1$ and n_2

4. If x, y, z are distinct positive reals such that $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$, then value of $x^x y^y z^z$ is

(a) 1 (b) 0

(c) -1 (d) none of these

5. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is

(a) less than $4ab$ (b) greater than $-4ab$

(c) less than $-4ab$ (d) greater than $4ab$

6. Solve for x :
$\log_{2x+3}(6x^2 + 23x + 21) + \log_{3x+7}(4x^2 + 12x + 9) = 4$

(a) -4 (b) -2

(c) $-\dfrac{1}{4}$ (d) All of these

One or More Than One Option(s) Correct Type

7. $ABCD$ is a rhombus, its diagonals AC and BD intersect at the point R where $BD = 2AC$. Its points D and R represent the complex numbers $1 + i$ and $2 - i$ respectively, then the complex number represented by A is

(a) $(3, -1/2)$ or $(1, -1/2)$

(b) $(3, -1/2)$ or $(1, -3/2)$

(c) $(-1/2, -3/2)$ or $(-3/2, -1/2)$

(d) None of these

8. $(1 + i)^5 + (1 - i)^5 =$

(a) -8 (b) 8

(c) $2^{7/2}\cos\dfrac{5\pi}{4}$ (d) $-2^{7/2}\cos\dfrac{5\pi}{4}$

9. If α and β are non-real cube roots of unity and $x = a + b$, $y = a\alpha + b\beta$, $z = a\beta + b\alpha$, then

(a) $x + y + z = 1$

(b) $x + y + z = 0$

(c) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$

(d) none of these

10. The equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \gamma\beta$, if α, β are the roots of the equation $ax^2 + bx + c = 0$ and γ, δ are roots of the equation $a'x^2 + b'x + c' = 0$, is

(a) $aa'x^2 + bb'x + cc' = 0$
(b) $ax^2 + (a + a' + b')x + bc' = 0$
(c) $aa'x^2 - bb'x + cc' = 0$
(d) none of these

11. If A and B are the points $(3, -1)$ and $(2, 1)$ respectively, then the locus of the points $P(z)$, $z = x + yi$, $x, y \in R$, such that $|z - 3 + i| = |z - 2 - i|$ is
(a) a circle containing A and B
(b) P is equidistant from A and B
(c) right bisector of segment joining A and B
(d) none of these

12. The value of $x : |x^2 + 2x - 8| + x - 2 = 0$ is
(a) -5 (b) -2
(c) 2 (d) -3

13. If $a \in C$ be such that $|a| = 1$, then the equation $\left(\frac{1+iz}{1-iz}\right)^4 = a$ has all the roots
(a) real and distinct
(b) non-real
(c) two real and two non-real
(d) none of these

Comprehension Type

Let α, β be the roots of the equation $6x^2 + 6px + p^2 = 0$, where p is a real number.

14. If both α and β are greater than 2, then
(a) $p < -4$
(b) $p < -2\sqrt{6}$
(c) $p < -6 - 2\sqrt{6}$
(d) none of these

15. The equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is
(a) $3x^2 + 4p^2x + p^4 = 0$
(b) $3x^2 - 4p^2x + p^4 = 0$
(c) $3x^2 - 4p^2x - p^4 = 0$
(d) none of these

Matrix Match Type

16. Match the following :

Column-I		Column-II	
P.	If a and b are positive numbers and $\log \frac{a+b}{2} = \frac{1}{2}(\log a + \log b)$ then $\frac{a}{b}$ is equal to	1.	$3\sqrt{3}$
Q.	Let $A(2 + 0i)$, $B(-1 + \sqrt{3}i)$ and $C(-1 - \sqrt{3}i)$ be the vertices of ΔABC. Then, 6 sin A is equal to	2.	1
R.	If one root of the equation $(x - 1)(7 - x) = \lambda$ is three times the other, then λ =	3.	5
S.	Conjugate of the complex number $-\frac{7}{2} - \frac{3\sqrt{3}i}{2}$ is	4.	$-2 + 3\omega$

	P	Q	R	S
(a)	4	2	3	1
(b)	3	4	1	2
(c)	1	4	3	2
(d)	2	1	3	4

Integer Answer Type

17. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is

18. The least integral value of k for which $(k - 2)x^2 + 8x + k + 4 > 0$ for all $x \in R$, is

19. If α, β are non-real cube roots of unity then $(1 + \alpha)(1 + \beta)(1 + \alpha^2)(1 + \beta^2)(1 + \alpha^4)(1 + \beta^4)$....upto $2n$ factors is equal to

20. If α be non real cube root of unity, then $\left|\sqrt{\alpha}\right|$ equals

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

ONCEPT
MAP

RELATIONS AND FUNCTIONS

Class XI

Cartesian Product of Sets

rtesian product of two sets A & B is denoted and fined as, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

rtesian product of two sets is not commutative.

te : (i) If $n(A) = p$, $n(B) = q$, then $n(A \times B) = p \times q$.

) $(a, b) = (p, q) \Leftrightarrow a = p$ & $b = q$.

i) Ordered triplet : $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$

Relations

- R is a relation from A to B (where $A, B \neq \phi$) if $R \subseteq A \times B \Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}$
- Dom $(R) = \{a : (a, b) \in R\}$
- Range $(R) = \{b : (a, b) \in R, \forall\ a \in A\}$
- If $n(A) = p$ and $n(B) = q$, then total number of relations is 2^{pq}.

Functions

elation $(f : A \to B)$ where every element of set A has y one image in set B.

m $(f) = \{a : (a, b) \in f\}$

domain $(f) = \{b : \forall\ b \in B\}$

nge $(f) = \{b : (a, b) \in f, \forall\ a \in A\}$

Algebra of Real Functions

For $f : X \to R$ and $g : X \to R$

- $(f + g)(x) = f(x) + g(x)$ for all $x \in X$
- $(f - g)(x) = f(x) - g(x)$ for all $x \in X$
- $(\alpha f)(x) = \alpha f(x)$ for all $x \in X$, α is a constant
- $(fg)(x) = f(x) \cdot g(x)$ for all $x \in X$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in X$

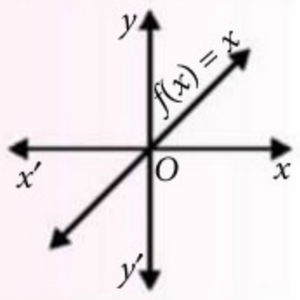
Types of Functions

ntity Function

nction $f : R \to R$ defined = x, $\forall\ x \in R$

n : R

: R



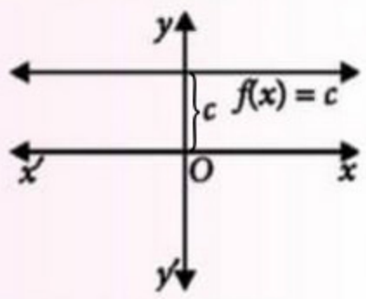
nstant Function

nction $f : R \to R$

d by $f(x) = c$, $\forall\ x \in R$

n : R

: $\{c\}$



ponential Function

ction f defined by

e^x as $e^x = f(x)$

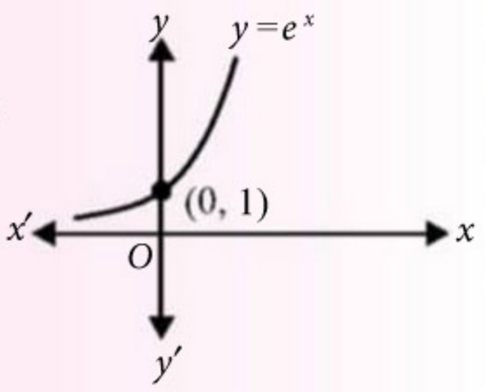
$x + \frac{x^2}{2!} + \frac{x}{3!} + \ldots\infty$

ed exponential

on

in : R

: $(0, \infty)$



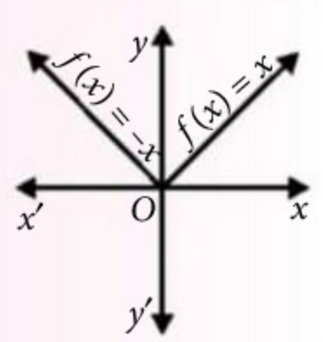
- **Modulus Function**

The function $f : R \to R$ defined by

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain : R

Range : $[0, \infty)$



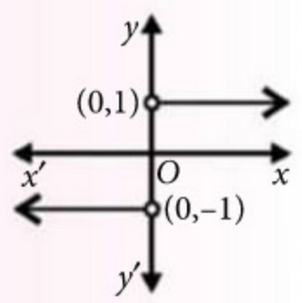
- **Signum Function**

The function $f : R \to R$ defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \\ 0, & x = 0 \end{cases}$$

Domain : R

Range : $\{-1, 0, 1\}$



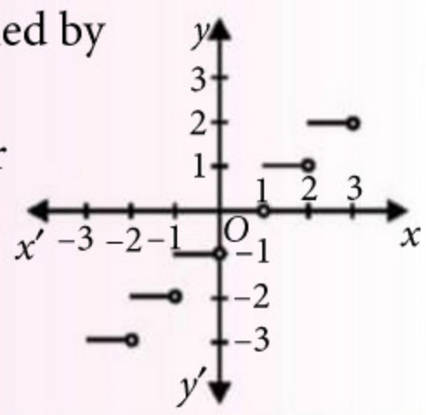
- **Greatest Integer Function**

The function $f : R \to R$ defined by

$$f(x) = [x] = \begin{cases} x, & x \in Z \\ \text{greatest integer} \\ \text{less than equal} \\ \text{to } x, & x \notin Z \end{cases}$$

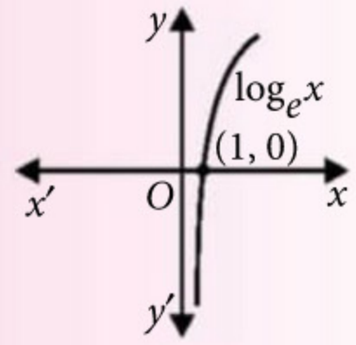
Domain : R

Range : Z



- **Logarithmic Function**

A function f defined by $f(x) = \log_e x$, $x > 0$ is called logarithmic function.



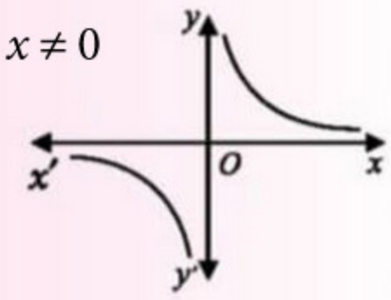
Domain : $(0, \infty)$

Range : R

- **Reciprocal Function**

A function f defined by

$f(x) = \frac{1}{x}$, $x \neq 0$ is called reciprocal function.



Domain : $R - \{0\}$

Range : $R - \{0\}$

DIFFERENTIABILITY

CONCEP MAP

Class XII

A real valued function $f(x)$ is differentiable at $x = c$ if $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ exists finitely or $Lf'(a) = Rf'(a)$

where $Lf'(a)$ = Left hand derivative = $\lim\limits_{h\to 0}\dfrac{f(c-h)-f(c)}{-h}$ and

$Rf'(a)$ = Right hand derivative = $\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$

Note : If a function is differentiable at a point then it is continuous at that point. But the converse is not always true.

Definition

- **Parametric Differentiation:**
  If $x = f(t)$ and $y = g(t)$ then,
  $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)},\ f'(t) \neq 0$
- **Logarithmic Differentiation:**
  If $y = u^v$, where u and v are functions of x, then apply log on both sides to get $\log y = v\log u$
  $\Rightarrow \dfrac{d}{dx}(u^v) = u^v\left[\dfrac{v}{u}\dfrac{du}{dx} + \log u\dfrac{dv}{dx}\right]$
- **Chain Rule:**
  If $y = f(t)$, $t = g(u)$ and $u = m(x)$
  then $\dfrac{dy}{dx} = \dfrac{dy}{dt}\times\dfrac{dt}{du}\times\dfrac{du}{dx}$

Some Standard Derivatives

- $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}(\cot^{-1}x) = \dfrac{-1}{1+x^2}$
- $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2-1}}$
- $\dfrac{d}{dx}(\operatorname{cosec}^{-1}x) = \dfrac{-1}{x\sqrt{x^2}}$
- $\dfrac{d}{dx}a^x = a^x\log a$
- $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$
- $\dfrac{d}{dx}\ln(x+\sqrt{x^2+1}) = \dfrac{1}{\sqrt{x^2+1}}$
- $\dfrac{d}{dx}\ln(x+\sqrt{x^2-1}) =$

Solving Methods

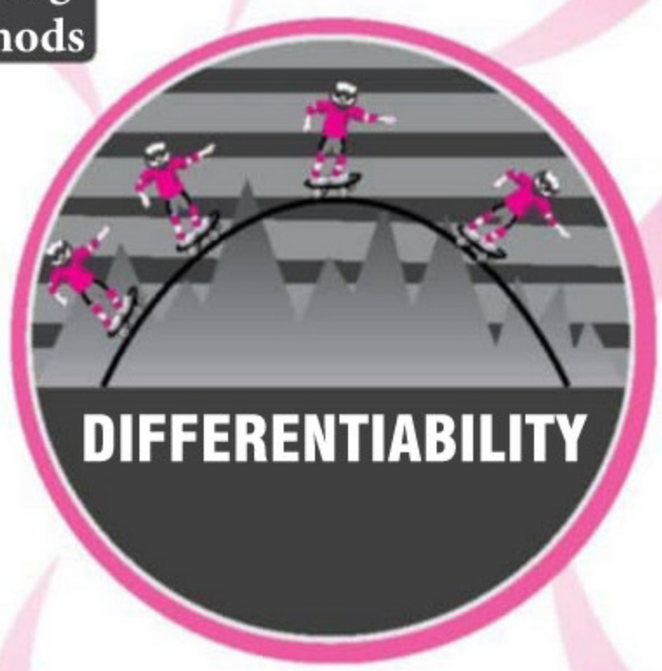
Second Order Derivative

Let $y = f(x)$, then $\dfrac{dy}{dx} = f'(x)$.

If $f'(x)$ is different then

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = f''(x)$

or $\dfrac{d^2y}{dx^2} = f''(x)$.



Formulae for Higher Order Derivatives

- $\dfrac{d^n}{dx^n}\ln(ax+b) = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
- $\dfrac{d^n}{dx^n}\sin(ax+b) = a^n\sin\left(ax+b+\dfrac{n\pi}{2}\right)$
- $\dfrac{d^n}{dx^n}\cos(ax+b) = a^n\cos\left(ax+b+\dfrac{n\pi}{2}\right)$
- $\dfrac{d^n}{dx^n}e^{ax}\sin(bx+c) = r^ne^{ax}\sin(bx+c+n\theta)$
- $\dfrac{d^n}{dx^n}e^{ax}\cos(bx+c) = r^ne^{ax}\cos(bx+c+n\theta)$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\dfrac{b}{a}$

Leibnitz Formula

The n^{th} derivative o product of two funct

If u, v be two func possessing derivati the n^{th} order, then

$(uv)_n = u_nv + {}^nC_1\,u_r$
$+ {}^nC_2\,u_{n-2}v_2 + \ldots.$
$u_{n-r}v_r + \ldots. + {}^nC_nuv_n$

$= \sum_{r=0}^{n} C(n,r)\,u_{n-r}v_r$ w

$C(n,r) = {}^nC_r = \dfrac{n!}{(n-r)!}$

and $C_0, C_1, \ldots\ldots, C_n$ ar binomial coefficient

CBSE warm-up!

CLASS-XII

Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2020-21.

Series 2 — Inverse Trigonometric Functions

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. Evaluate : $\cos^{-1}(\cos(-680°))$

(a) $40°$ (b) $60°$ (c) $35°$ (d) $90°$

2. Find the value of $\cos^2\left(\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\right)$.

(a) $\frac{1}{4}$ (b) $\frac{4}{5}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

3. Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$.

(a) π (b) $\frac{3\pi}{2}$
(c) $\frac{2\pi}{3}$ (d) None of these

4. Find the domain of $\text{cosec}^{-1}(3-2x)$.

(a) $(-\infty, 2]$ (b) $(-\infty, 1] \cup [-2, \infty)$
(c) $[1, \infty)$ (d) $[1, 2]$

5. Find the principal value of $\cos^{-1}\left\{\sin\left(\cos^{-1}\frac{1}{2}\right)\right\}$.

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

6. Find the principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

7. If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then find the value of $\cot^{-1}x + \cot^{-1}y$.

(a) $\frac{4\pi}{5}$ (b) $\frac{3\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{5}$

8. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

9. If $\tan^{-1}\left(\sqrt{3}\right) + \cot^{-1}(x) = \frac{\pi}{2}$, then find x.

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{2}$ (d) $2\sqrt{3}$

10. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$, $xy<1$, then write the value of $x+y+xy$.

(a) 5 (b) 4 (c) 1 (d) 0

(Q. 11-Q.15) Fill in the blanks.

11. The principal value of

$\tan^{-1}1+\cos^{-1}\left(\frac{-1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)$ is _____.

12. The principal value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$ is_____.

OR

The value of $\text{cosec}^{-1}(2/\sqrt{3})$ is _____.

13. $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to _____.

14. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then the value of x is _____.

OR

The value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ is _____.

15. If $\cot^{-1}x+\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)=\frac{\pi}{4}$, then x is equal to ___.

(Q. 16-Q.20) Answer the following questions.

16. Find the value of $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)$.

17. Evaluate : $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right\}$

OR

Prove that : $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x\in[0,1]$

18. If $\tan^{-1}\frac{4}{3}=\theta$, find the value of $\cos\theta$.

19. Write $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$ in simplest form.

20. For the principal values, evaluate the following:

$\cot^{-1}(-\sqrt{3})+\tan^{-1}(1)+\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

SECTION - B

21. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that $x^2+y^2+z^2+2xyz=1$.

22. Solve for x : $\cos(\tan^{-1}x)=\sin\left(\sec^{-1}\frac{13}{12}\right)$.

23. If in a ΔABC, $\angle A=\tan^{-1}2$ and $\angle B=\tan^{-1}3$, then show that $\angle C$ is equal to $\frac{\pi}{4}$.

OR

Evaluate : $\sin\left(2\cot^{-1}\left(-\frac{5}{12}\right)\right)$.

24. Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{36}{85}$.

25. Solve : $\sin^{-1}x=\frac{\pi}{6}+\cos^{-1}x$

OR

Prove that : $4\,(\cot^{-1}3+\text{cosec}^{-1}\sqrt{5})=\pi$.

26. Prove that : $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$.

SECTION - C

27. Write each of the following in the simplest form:

(i) $\tan^{-1}(\sec x+\tan x)$ **(ii)** $\sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

28. Evaluate : $\tan\left[\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right]$, $0\le 2\theta\le\pi$

OR

Solve : $2\tan^{-1}(\cos x)=\tan^{-1}(2\text{cosec}\,x)$.

29. Prove that $\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{2}$, where $x, y, z>0$ such that $x^2+y^2+z^2=r^2$.

30. Solve for x : $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$

31. If $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}=\alpha$, then prove that $x^2=\sin 2\alpha$.

OR

If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, prove that

$\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

32. Prove that :

$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, \frac{-1}{\sqrt{2}}\le x\le 1$

SECTION - D

33. Solve :

(i) $\cos\left(\sin^{-1}\frac{1}{3}+\cos^{-1}x\right)=0$

(ii) $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\pi+\tan^{-1}(-7)$

OR

If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then find x.

34. Prove the following :

$$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}$$

35. Prove that

(i) $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

(ii) $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$

OR

If $\tan^{-1}\left(\frac{1}{1+1\cdot2}\right)+\tan^{-1}\left(\frac{1}{1+2\cdot3}\right)$

$$+\ldots+\tan^{-1}\left(\frac{1}{1+n\cdot(n+1)}\right)=\tan^{-1}\theta,$$

then find the value of θ.

36. Prove that :

$$\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$$

$$=(\alpha+\beta)(\alpha^2+\beta^2).$$

SOLUTIONS

1. (a) : $\cos^{-1}\{\cos(-680°)\}=\cos^{-1}(\cos 680°)$

$$=\cos^{-1}\left(\cos\frac{34\pi}{9}\right)=\cos^{-1}\left\{\cos\left(4\pi-\frac{2\pi}{9}\right)\right\}$$

$$=\cos^{-1}\left(\cos\frac{2\pi}{9}\right)=\frac{2\pi}{9}=40°$$

2. (b) : Let $\cos^{-1}\frac{3}{5}=\theta\Rightarrow\cos\theta=\frac{3}{5}$

$$\therefore\quad\cos^2\left\{\left(\frac{1}{2}\right)\cos^{-1}\left(\frac{3}{5}\right)\right\}=\cos^2\left(\frac{\theta}{2}\right)$$

$$=\frac{\cos\theta+1}{2}=\frac{\frac{3}{5}+1}{2}=\frac{4}{5}$$

3. (c) : We have, $\cos^{-1}\left(\frac{1}{2}\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$

Also, $\sin^{-1}\left(\frac{1}{2}\right)=\sin^{-1}\left(\sin\frac{\pi}{6}\right)=\frac{\pi}{6}$

$$\therefore\quad\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+\left(2\times\frac{\pi}{6}\right)=\frac{2\pi}{3}$$

4. (b) : For the domain of $\operatorname{cosec}^{-1}(3-2x)$, $|3-2x|\geq1$

$\Rightarrow\quad 3-2x\leq-1$ or $3-2x\geq1$

$\Rightarrow\quad -2x\leq-4$ or $-2x\geq-2\quad\Rightarrow\quad x\geq2$ or $x\leq1$.

$\therefore\quad$ Domain of $\operatorname{cosec}^{-1}(3-2x)$ is $(-\infty,1]\cup[2,\infty)$.

5. (a) : We know that $\cos^{-1}\frac{1}{2}=\frac{\pi}{3}$.

$$\therefore\quad\cos^{-1}\left\{\sin\left(\cos^{-1}\frac{1}{2}\right)\right\}=\cos^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}\qquad\left[\because\quad\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}\right]$$

6. (a) : Since, $\sin\frac{2\pi}{3}=\sin\left(\pi-\frac{\pi}{3}\right)=\sin\frac{\pi}{3}$

$$\therefore\quad\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}.$$

Hence, the principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is $\frac{\pi}{3}$.

7. (d) : We have, $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$

$$\Rightarrow\quad\left(\frac{\pi}{2}-\cot^{-1}x\right)+\left(\frac{\pi}{2}-\cot^{-1}y\right)=\frac{4\pi}{5}$$

$$\left[\because\quad\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$$

$$\Rightarrow\quad\pi-(\cot^{-1}x+\cot^{-1}y)=\frac{4\pi}{5}$$

$$\Rightarrow\quad\cot^{-1}x+\cot^{-1}y=\pi-\frac{4\pi}{5}=\frac{\pi}{5}$$

8. (b) : $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$$=\tan^{-1}\left[2\sin\left(2.\frac{\pi}{6}\right)\right]=\tan^{-1}\left[2\sin\frac{\pi}{3}\right]$$

$$=\tan^{-1}.\left[2.\frac{\sqrt{3}}{2}\right]=\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}.$$

9. (a) : $\tan^{-1}\left(\sqrt{3}\right)+\cot^{-1}(x)=\frac{\pi}{2}$

$$\Rightarrow\quad\frac{\pi}{3}+\cot^{-1}x=\frac{\pi}{2}\Rightarrow\cot^{-1}x=\frac{\pi}{6}$$

$\Rightarrow \quad x = \cot\frac{\pi}{6} \Rightarrow x = \sqrt{3}$

10. (c) : Given: $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4} \qquad (xy < 1)$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4} \Rightarrow \quad \frac{x+y}{1-xy} = \tan\frac{\pi}{4} = 1$

$\Rightarrow \quad x + y = 1 - xy \Rightarrow x + y + xy = 1$

11. Principal value of $\tan^{-1}(1)$ is $\frac{\pi}{4}$.

Principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is $\frac{2\pi}{3}$.

Also, principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$ is $\left(\frac{-\pi}{6}\right)$.

$\therefore$ Principal value of $\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$

$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi}{4}$

12. Principal value of $\cos^{-1}\frac{1}{2}$ is $\frac{\pi}{3}$.

Principal value of $\sin^{-1}\frac{1}{2}$ is $\frac{\pi}{6}$.

So, the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2\times\frac{\pi}{6}$

$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

OR

Let $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta \Rightarrow \quad \operatorname{cosec}\theta = \frac{2}{\sqrt{3}} \Rightarrow \quad \theta = \frac{\pi}{3}$

13. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \tan^{-1}\left\{\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right\}$

$= \tan^{-1}\left\{\frac{9+8}{36-2}\right\} = \tan^{-1}\left(\frac{1}{2}\right)$

14. $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

or $\quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}1$

$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{2}$

$\left[\because \quad \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1}x \quad \Rightarrow \quad x = \sin\left(\sin^{-1}\frac{1}{5}\right) = \frac{1}{5}$

OR

Since $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, for $|x| < 1$

So, $2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)$

$= \tan^{-1}\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right) = \tan^{-1}\left(\frac{5}{12}\right)$

$\therefore \quad \tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{12}$

15. $\because \quad \cot^{-1}x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{2} = \tan^{-1}1$

$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1}1 - \tan^{-1}\frac{1}{2} = \tan^{-1}\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)$

$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3} \Rightarrow x = 3$

16. $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = \cot\left[\frac{\pi}{4} - \cot^{-1}\left(\frac{3^2-1}{2\times3}\right)\right]$

$= \cot\left(\frac{\pi}{4} - \cot^{-1}\frac{4}{3}\right) = \frac{\cot\frac{\pi}{4}\cdot\frac{4}{3}+1}{\frac{4}{3}-\cot\frac{\pi}{4}} = \frac{\frac{4}{3}+1}{\frac{4}{3}-1} = 7$

17. We know that, $\sin^{-1}(-\theta) = -\sin^{-1}\theta$, for $\theta \in [-1, 1]$

$\therefore \quad \sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\} = \sin\left\{\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right\}$

$= \sin\left(\frac{\pi}{3} + \sin^{-1}\left(\sin\frac{\pi}{6}\right)\right) \quad = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1.$

OR

$\tan^{-1}\sqrt{x} = \frac{1}{2}(2\tan^{-1}\sqrt{x}) = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

18. Given, $\tan^{-1}\frac{4}{3} = \theta$, where $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \tan\theta = \frac{4}{3}.$

We know that $\cos\theta > 0$, when $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+\frac{16}{9}}} = \frac{3}{5}$$

19. Put $x = a\tan y \Rightarrow y = \tan^{-1}\frac{x}{a}$.

$$\therefore \quad \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}\left(\frac{3a^2 \cdot a\tan y - a^3\tan^3 y}{a^3 - 3a\cdot a^2\tan^2 y}\right)$$

$$= \tan^{-1}\left(\frac{3\tan y - \tan^3 y}{1 - 3\tan^2 y}\right)$$

$$= \tan^{-1}(\tan 3y) = 3y = 3\tan^{-1}\frac{x}{a}.$$

20. $\cot^{-1}(-\sqrt{3}) + \tan^{-1}(1) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$$= \cot^{-1}\cot\left(\frac{5\pi}{6}\right) + \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \sec^{-1}\left(\sec\frac{\pi}{6}\right)$$

$$= \frac{5\pi}{6} + \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{4}$$

21. Let $\cos^{-1}x = \alpha$, $\cos^{-1}y = \beta$ and $\cos^{-1}z = \gamma$

$\Rightarrow \quad \cos\alpha = x, \cos\beta = y$ and $\cos\gamma = z$

From question, $\alpha + \beta + \gamma = \pi \quad \therefore \quad \alpha + \beta = \pi - \gamma$

$\therefore \quad \cos(\alpha + \beta) = \cos(\pi - \gamma)$

$\Rightarrow \quad \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\cos\gamma$

$$\Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$$

$$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$$

Squaring, we get

$x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

or $\quad x^2 + y^2 + z^2 + 2xyz = 1.$

22. Let $\tan^{-1}x = \phi$. Then, $\tan\phi = \frac{x}{1}$

$$\therefore \quad \cos\phi = \frac{1}{\sqrt{1+x^2}} \Rightarrow \phi = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

So, $\cos(\tan^{-1}x) = \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right) = \frac{1}{\sqrt{1+x^2}}$

Also, let $\sec^{-1}\frac{13}{12} = \theta$. Then, $\sec\theta = \frac{13}{12}$

$$\therefore \quad \sin\theta = \frac{5}{13} \Rightarrow \theta = \sin^{-1}\frac{5}{13}$$

So, $\sin\left(\sec^{-1}\frac{13}{12}\right) = \sin\left(\sin^{-1}\frac{5}{13}\right) = \frac{5}{13}$

Thus, $\frac{1}{\sqrt{1+x^2}} = \frac{5}{13} \quad \Rightarrow \quad \frac{1}{(1+x^2)} = \frac{25}{169}$

$$\Rightarrow \quad 1 + x^2 = \frac{169}{25} \Rightarrow x^2 = \frac{169}{25} - 1 \Rightarrow x = \pm\frac{144}{25}$$

Hence, $x = \pm\frac{12}{5}$.

23. We have, $\angle A = \tan^{-1}2$, $\angle B = \tan^{-1}3$

We know that, $\angle A + \angle B + \angle C = \pi$

$\Rightarrow \quad \tan^{-1}2 + \tan^{-1}3 + \angle C = \pi$

$$\Rightarrow \quad \pi + \tan^{-1}\left(\frac{2+3}{1-2\times3}\right) + \angle C = \pi$$

$$\Rightarrow \quad \pi + \tan^{-1}(-1) + \angle C = \pi \Rightarrow \pi - \frac{\pi}{4} + \angle C = \pi$$

$$\Rightarrow \quad \frac{3\pi}{4} + \angle C = \pi \Rightarrow \angle C = \frac{\pi}{4}$$

OR

Let $\cot^{-1}\left(-\frac{5}{12}\right) = x, 0 < x < \pi$

$\Rightarrow \quad \cot x = -\frac{5}{12}, 0 < x < \pi.$

For these values of x, $\operatorname{cosec} x > 0$.

$$\therefore \quad \operatorname{cosec} x = \sqrt{1+\cot^2 x} = \sqrt{1+\left(-\frac{5}{12}\right)^2} = \frac{13}{12}$$

$$\Rightarrow \quad \sin x = \frac{12}{13}.$$

Now, $\cos x = \frac{\cos x}{\sin x}\cdot\sin x = \cot x\cdot\sin x$

$$= \left(-\frac{5}{12}\right)\cdot\frac{12}{13} = -\frac{5}{13}.$$

$$\therefore \quad \sin\left(2\cot^{-1}\left(-\frac{5}{12}\right)\right) = \sin 2x = 2\sin x\cos x = -\frac{120}{169}.$$

24. L.H.S. $= \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17} + \cos^{-1}\frac{4}{5}$

$$= \cos^{-1}\left\{\left(\frac{15}{17}\times\frac{4}{5}\right) - \sqrt{1-\left(\frac{15}{17}\right)^2}\cdot\sqrt{1-\left(\frac{4}{5}\right)^2}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \sqrt{\frac{64}{289}}\cdot\sqrt{\frac{9}{25}}\right\} = \cos^{-1}\left\{\frac{12}{17} - \frac{8}{17}\times\frac{3}{5}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \frac{24}{85}\right\} = \cos^{-1}\left(\frac{36}{85}\right) = \text{R.H.S.}$$

Hence, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{36}{85}$.

25. We have, $\sin^{-1}x = \frac{\pi}{6} + \cos^{-1}x$

$$\Rightarrow \quad \sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow \quad \sin^{-1}x-\left(\frac{\pi}{2}-\sin^{-1}x\right)=\frac{\pi}{6}$$

$$\Rightarrow \quad 2\sin^{-1}x=\frac{2\pi}{3} \Rightarrow \sin^{-1}x=\frac{\pi}{3} \Rightarrow x=\frac{\sqrt{3}}{2}$$

OR

L.H.S. $= 4(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5})$

$$=4\left(\tan^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$$

$$=4\left[\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left\{\frac{(1/\sqrt{5})}{\sqrt{1-(1/\sqrt{5})^2}}\right\}\right]$$

$$=4\left[\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right]$$

$$=4\tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}\right]=4\tan^{-1}\left(\frac{5}{5}\right)$$

$$=4\tan^{-1}(1)=4\times\frac{\pi}{4}=\pi=\text{R.H.S.}$$

26. We have, L.H.S.

$$=\left\{\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)\right\}-\tan^{-1}\left(\frac{8}{19}\right)$$

$$=\tan^{-1}\left\{\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right\}-\tan^{-1}\left(\frac{8}{19}\right)$$

$$=\tan^{-1}\left(\frac{27}{11}\right)-\tan^{-1}\left(\frac{8}{19}\right)$$

$$=\tan^{-1}\left\{\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right\}=\tan^{-1}\left(\frac{425}{425}\right)$$

$$=\tan^{-1}1=\frac{\pi}{4}=\text{R.H.S.}$$

27. (i) $\sec x+\tan x=\dfrac{1+\sin x}{\cos x}=\dfrac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}$

$$=\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$\therefore \quad \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)=\frac{\pi}{4}+\frac{x}{2}$$

(ii) Put $x = \cos\theta$

$$\therefore \quad \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)=\sin\left(2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$$

$$=\sin\left(2\tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right)$$

$$=\sin\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right]=\sin\theta=\sqrt{1-x^2}$$

28. Let $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)=2\theta \Rightarrow \cos 2\theta=\frac{\sqrt{5}}{3}$ and $0\le 2\theta\le\pi$

Now, $\cos 2\theta=\dfrac{\sqrt{5}}{3}$

$$\therefore \quad \frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{\sqrt{5}}{3} \Rightarrow \frac{2}{2\tan^2\theta}=\frac{3+\sqrt{5}}{3-\sqrt{5}}$$

[By componendo and dividendo]

$$\Rightarrow \quad \tan^2\theta=\left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{3-\sqrt{5}}\right)$$

$$\Rightarrow \quad \tan^2\theta=\frac{(3-\sqrt{5})^2}{4} \Rightarrow \tan\theta=\pm\left(\frac{3-\sqrt{5}}{2}\right)$$

But $0\le 2\theta\le\pi \ \therefore \ 0\le\theta\le\dfrac{\pi}{2}$

Hence $\tan\theta$ must be positive.

$$\therefore \quad \tan\theta=\frac{3-\sqrt{5}}{2}.$$

OR

Given equation is $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$ ...(i)

Let $2\tan^{-1}(\cos x) = \theta \Rightarrow \cos x = \tan(\theta/2)$ ...(ii)

Now, $2\operatorname{cosec} x=\dfrac{2}{\sin x}=\dfrac{2}{\sqrt{1-\cos^2 x}}=\dfrac{2}{\sqrt{1-\tan^2(\theta/2)}}$ ...(iii)

From (ii) and (iii), given equation becomes

$$\theta=\tan^{-1}\left[\frac{2}{\sqrt{1-\tan^2\left(\frac{\theta}{2}\right)}}\right]\Rightarrow\tan\theta=\frac{2}{\sqrt{1-\tan^2\left(\frac{\theta}{2}\right)}}$$

$$\Rightarrow \quad \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}=\frac{2}{\sqrt{1-\tan^2(\theta/2)}}$$

$$\Rightarrow \quad \tan(\theta/2)=\sqrt{1-\tan^2(\theta/2)} \qquad \text{...(iv)}$$

$$\Rightarrow \quad \tan^2(\theta/2) = 1 - \tan^2(\theta/2) \Rightarrow 2\tan^2(\theta/2) = 1$$

$$\Rightarrow \quad \tan\frac{\theta}{2}=\frac{1}{\sqrt{2}} \qquad \left[\because \text{ From (iv), } \tan\frac{\theta}{2}>0\right]$$

$\Rightarrow \cos x = \frac{1}{\sqrt{2}}$ [From (ii)]

$\Rightarrow \cos x = \cos\frac{\pi}{4} \quad \therefore \quad x = 2n\pi \pm \frac{\pi}{4}, n \in Z$

But for equation (i) to be satisfied cosec x and cos x must have same sign.

$\therefore$ x lies in 1st quadrant.

$\therefore \quad x = 2n\pi + \frac{\pi}{4}, n \in Z$

29. We have, $x^2 + y^2 + z^2 = r^2$

Also, $\frac{yz}{xr} \times \frac{zx}{yr} = \frac{z^2}{r^2} = \frac{z^2}{x^2+y^2+z^2} < 1$

$\therefore \quad \tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{xz}{yr} + \tan^{-1}\frac{xy}{zr}$

$$= \tan^{-1}\left\{\frac{\frac{yz}{xr}+\frac{xz}{yr}}{1-\frac{yz}{xr}\times\frac{zx}{yr}}\right\} + \tan^{-1}\frac{xy}{zr}$$

$$= \tan^{-1}\left\{\frac{\frac{z(x^2+y^2)}{xyr}}{1-\frac{z^2}{r^2}}\right\} + \tan^{-1}\left(\frac{xy}{zr}\right)$$

$$= \tan^{-1}\left\{\frac{z(x^2+y^2)}{xyr}\times\frac{r^2}{x^2+y^2}\right\} + \tan^{-1}\left(\frac{xy}{zr}\right)$$

$$= \tan^{-1}\left(\frac{zr}{xy}\right) + \tan^{-1}\left(\frac{xy}{zr}\right)$$

$$= \cot^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{xy}{zr}\right) = \frac{\pi}{2}$$

30. We have, $\sin^{-1} x + \sin^{-1}(1 - x) = \cos^{-1} x$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1}(1-x) = \frac{\pi}{2} - \sin^{-1} x$

$\Rightarrow \quad \sin^{-1}(1-x) = \frac{\pi}{2} - 2\sin^{-1} x$

$\Rightarrow \quad \sin\left\{\sin^{-1}(1-x)\right\} = \sin\left(\frac{\pi}{2} - 2\sin^{-1} x\right)$

$\Rightarrow \quad 1 - x = \cos(2\sin^{-1} x)$

$\Rightarrow \quad 1 - x = \cos\{\cos^{-1}(1 - 2x^2)\}$

$[\because 2\sin^{-1} x = \cos^{-1}(1 - 2x^2)$

$\Rightarrow \quad 1 - x = 1 - 2x^2 \Rightarrow 2x^2 - x = 0 \Rightarrow x(2x - 1) = 0$

$\Rightarrow \quad x = 0$ or, $x = \frac{1}{2}$

Clearly, these values satisfy the given equation.

Hence, $x = 0, \frac{1}{2}$ are the roots of the given equation.

31. We have, $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\} = \alpha$

$$\Rightarrow \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \tan\alpha$$

$$\Rightarrow \frac{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)+\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)-\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)} = \frac{\tan\alpha+1}{\tan\alpha-1}$$

$$\Rightarrow \frac{2\sqrt{1+x^2}}{-2\sqrt{1-x^2}} = \frac{\tan\alpha+1}{\tan\alpha-1} \Rightarrow \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}} = \frac{1-\tan\alpha}{1+\tan\alpha}$$

$$\Rightarrow \sqrt{\frac{1-x^2}{1+x^2}} = \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}$$

$$\Rightarrow \frac{1-x^2}{1+x^2} = \frac{1-\sin 2\alpha}{1+\sin 2\alpha} \Rightarrow x^2 = \sin 2\alpha$$

OR

We have, $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$

$$\Rightarrow \cos^{-1}\left[\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right] = \alpha$$

$$\Rightarrow \frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha \quad \ldots (1)$$

Squaring on both sides, we get

$$\frac{x^2y^2}{a^2b^2} + \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right) - \frac{2xy}{ab}\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos^2\alpha$$

$$\Rightarrow \frac{x^2y^2}{a^2b^2} + 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} + \frac{x^2y^2}{a^2b^2} - \frac{2xy}{ab}\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = 1 - \sin^2\alpha$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\left[\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right] = \sin^2\alpha$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha \qquad \text{[From (1)]}$$

32. Putting $x = \cos\theta$, we get

$$\text{L.H.S.} = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2(\theta/2)}-\sqrt{2\sin^2(\theta/2)}}{\sqrt{2\cos^2(\theta/2)}+\sqrt{2\sin^2(\theta/2)}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos(\theta/2)-\sin(\theta/2)}{\cos(\theta/2)+\sin(\theta/2)}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}\right\}$$

[Dividing numerator and denominator by $\cos(\theta/2)$]

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} = \frac{\pi}{4}-\frac{\theta}{2}$$

$$= \left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right) = \text{R.H.S.}$$

33. **(i)** Here, $\cos\left(\sin^{-1}\frac{1}{3}+\cos^{-1}x\right) = 0 = \cos\left(\pm\frac{\pi}{2}\right)$

$\therefore \quad \sin^{-1}\frac{1}{3}+\cos^{-1}x = \pm\frac{\pi}{2}$

i.e., $\cos^{-1}x = \pm\frac{\pi}{2}-\sin^{-1}\frac{1}{3}$

$\therefore \quad x = \cos\left(\pm\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$

$x = \cos\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$	$x = \cos\left(-\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$
$= \sin\left(\sin^{-1}\frac{1}{3}\right) = \frac{1}{3}$	$= \cos\left(\frac{\pi}{2}+\sin^{-1}\frac{1}{3}\right)$
	$= -\sin\left(\sin^{-1}\frac{1}{3}\right) = -\frac{1}{3}$

$\therefore \quad x = \pm\frac{1}{3}$

(ii) We have,

$$\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right) = \pi+\tan^{-1}(-7)$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\frac{x-1}{x}}\right) = \pi+\tan^{-1}(-7)$$

$$\therefore \quad \frac{(x+1)x+(x-1)^2}{(x-1)x-(x+1)(x-1)} = \tan\left[\pi+\tan^{-1}(-7)\right]$$

$\Rightarrow \quad x^2 + x + x^2 - 2x + 1 = -7(x^2 - x - x^2 + 1)$

$\Rightarrow \quad 2x^2 - 8x + 8 = 0 \quad i.e., \quad x^2 - 4x + 4 = 0$

$\Rightarrow \quad (x-2)^2 = 0$

$\therefore \quad x = 2$

OR

$$(\tan^{-1}x)^2+(\cot^{-1}x)^2 = \frac{5\pi^2}{8} \qquad \text{[Given]}$$

$$\Rightarrow \quad (\tan^{-1}x)^2+\left(\frac{\pi}{2}-\tan^{-1}x\right)^2 = \frac{5\pi^2}{8}$$

Putting $\tan^{-1}x = \theta$, we get

$$\theta^2+\left(\frac{\pi}{2}-\theta\right)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \theta^2+\frac{\pi^2}{4}+\theta^2-\pi\theta = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad 2\theta^2-\pi\theta+\left(\frac{\pi^2}{4}-\frac{5\pi^2}{8}\right) = 0$$

$$\Rightarrow \quad 2\theta^2-\pi\theta-\frac{3}{8}\pi^2 = 0$$

$\Rightarrow \quad 16\theta^2 - 8\pi\theta - 3\pi^2 = 0 \Rightarrow 4\theta(4\theta - 3\pi) + \pi(4\theta - 3\pi) = 0$

$\Rightarrow \quad (4\theta + \pi)(4\theta - 3\pi) = 0 \Rightarrow$ Either $4\theta = 3\pi$ or $4\theta = -\pi$

$$\Rightarrow \quad \theta = \frac{3\pi}{4} \quad \text{or} \quad \theta = -\frac{\pi}{4}$$

Hence, $\tan^{-1}x = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$

$\Rightarrow \quad x = -1$

34. Let $\sin^{-1}\frac{3}{5} = \theta$ and $\cot^{-1}\frac{3}{2} = \phi$

$$\Rightarrow \quad \sin\theta = \frac{3}{5} \text{ and } \cot\phi = \frac{3}{2}$$

$$\Rightarrow \quad \tan\theta = \frac{3}{4} \text{ and } \tan\phi = \frac{2}{3}$$

$$\therefore \quad \theta = \tan^{-1}\frac{3}{4} \text{ and } \phi = \tan^{-1}\frac{2}{3}$$

Thus, $\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}$

$$= \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right) = \tan^{-1}\left[\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right]$$

$$= \tan^{-1}\left(\frac{17}{6}\right) = \alpha \text{ (say)} \qquad \ldots(1)$$

$$\Rightarrow \quad \tan\alpha = \frac{17}{6} \Rightarrow \cos\alpha = \frac{6}{\sqrt{6^2+17^2}} = \frac{6}{5\sqrt{13}}$$

Now, L.H.S. $= \cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

$= \cos\alpha = \dfrac{6}{5\sqrt{13}} = \text{R.H.S.}$ [Using (1)]

35. **(i)** $\text{L.H.S.} = \sin^{-1}\dfrac{12}{13} + \cos^{-1}\dfrac{4}{5} + \tan^{-1}\dfrac{63}{16}$

$= \tan^{-1}\dfrac{12}{5} + \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{63}{16}$

$= \pi + \tan^{-1}\left(\dfrac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right) + \tan^{-1}\dfrac{63}{16}$

$= \pi + \tan^{-1}\left(-\dfrac{63}{16}\right) + \tan^{-1}\dfrac{63}{16}$

$= \pi - \tan^{-1}\dfrac{63}{16} + \tan^{-1}\dfrac{63}{16} = \pi$

(ii) Let $\cos^{-1}\dfrac{4}{5} = \theta$. Then $\cos\theta = \dfrac{4}{5}$.

$\therefore \quad \tan\theta = \dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \dfrac{3}{4} \Rightarrow \theta = \tan^{-1}\dfrac{3}{4}$

Consequently, $\cos^{-1}\dfrac{4}{5} = \tan^{-1}\dfrac{3}{4}$

$\therefore \quad \cos^{-1}\dfrac{4}{5} + \tan^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{3}{5}$

$= \tan^{-1}\left(\dfrac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\cdot\frac{3}{5}}\right) = \tan^{-1}\dfrac{27}{11}$

OR

$\tan^{-1}\left(\dfrac{1}{1+1\cdot 2}\right) + \tan^{-1}\left(\dfrac{1}{1+2\cdot 3}\right) + \ldots$

$\ldots + \tan^{-1}\left(\dfrac{1}{1+n\cdot(n+1)}\right) = \tan^{-1}\theta$

$\Rightarrow \tan^{-1}\left(\dfrac{2-1}{1+1\cdot 2}\right) + \tan^{-1}\left(\dfrac{3-2}{1+2\cdot 3}\right)$

$+ \ldots + \tan^{-1}\left(\dfrac{(n+1)-n}{1+n(n+1)}\right) = \tan^{-1}\theta$

$\Rightarrow \tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 +$

$+ \ldots + \tan^{-1}(n+1) - \tan^{-1}(n) = \tan^{-1}\theta$

$\Rightarrow \tan^{-1}(n+1) - \tan^{-1}1 = \tan^{-1}\theta$

$\Rightarrow \tan^{-1}\left(\dfrac{(n+1)-1}{1+(n+1)(1)}\right) = \tan^{-1}\theta$

$\Rightarrow \tan^{-1}\left(\dfrac{n}{n+2}\right) = \tan^{-1}\theta \Rightarrow \dfrac{n}{n+2} = \theta$

36. We have,

$\text{L.H.S.} = \dfrac{\alpha^3}{2}\operatorname{cosec}^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}\right) + \dfrac{\beta^3}{2}\sec^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}\right)$

$\Rightarrow \text{L.H.S.} = \dfrac{\alpha^3}{2\sin^2\theta} + \dfrac{\beta^3}{2\cos^2\phi},$

where $\theta = \dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}$ and $\phi = \dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}$

$\therefore \quad \text{L.H.S.} = \dfrac{\alpha^3}{1-\cos 2\theta} + \dfrac{\beta^3}{1+\cos 2\phi}$

$= \dfrac{\alpha^3}{1-\cos(\tan^{-1}\alpha/\beta)} + \dfrac{\beta^3}{1+\cos(\tan^{-1}\beta/\alpha)}$

$= \dfrac{\alpha^3}{1-\cos\left(\cos^{-1}\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)} + \dfrac{\beta^3}{1+\cos\left(\cos^{-1}\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)}$

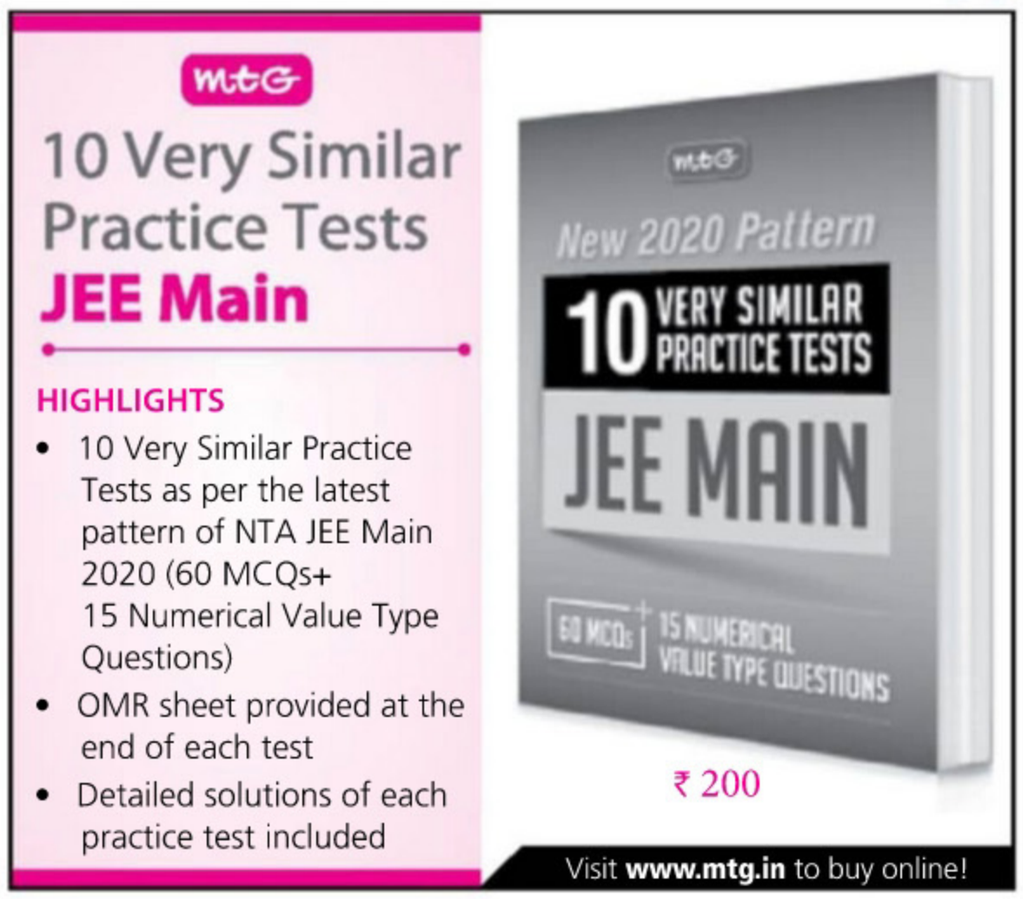
$= \dfrac{\alpha^3}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}} + \dfrac{\beta^3}{1+\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}$

$= \left\{\dfrac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta} + \dfrac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\alpha}\right\}\sqrt{\alpha^2+\beta^2}$

$= \left[\dfrac{\alpha^3\left\{\sqrt{\alpha^2+\beta^2}+\beta\right\}}{\alpha^2+\beta^2-\beta^2} + \dfrac{\beta^3\left\{\sqrt{\alpha^2+\beta^2}-\alpha\right\}}{\alpha^2+\beta^2-\alpha^2}\right]\sqrt{\alpha^2+\beta^2}$

$= \left\{\alpha\left(\sqrt{\alpha^2+\beta^2}+\beta\right) + \beta\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right\}\sqrt{\alpha^2+\beta^2}$

$= \alpha(\alpha^2+\beta^2) + \beta(\alpha^2+\beta^2) = (\alpha+\beta)(\alpha^2+\beta^2) = \text{R.H.S.}$



MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Series 2 : Inverse Trigonometric Functions

Only One Option Correct Type

1. The number of positive integral solutions of

$$\tan^{-1}x+\cos^{-1}\frac{y}{\sqrt{1+y^2}}=\sin^{-1}\frac{3}{\sqrt{10}}$$ is

(a) 0 (b) 1 (c) 2 (d) > 2

2. If $\left|\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right|<\frac{\pi}{3}$, then x belongs to the interval

(a) $\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$ (b) $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

(c) $\left(0,\frac{1}{\sqrt{3}}\right)$ (d) none of these

3. $\cos^{-1}\left[\cos\left(-\frac{17\pi}{15}\right)\right]$ is equal to

(a) $\frac{17\pi}{15}$ (b) $\frac{13\pi}{15}$

(c) $\frac{3\pi}{15}$ (d) $-\frac{17\pi}{15}$

4. The domain of the function

$$f(x)=\cos^{-1}\left(\frac{2-|x|}{4}\right)+[\log(3-x)]^{-1}$$ is

(a) $[-6, 6]$ (b) $[-5, 2)\cup(2, 3)$

(c) $(2, 3)$ (d) $[-6, 2)\cup(2, 3)$

5. If θ are ϕ are the roots of the equation $8x^2 + 22x + 5 = 0$, then

(a) both $\sin^{-1}\theta$ and $\sin^{-1}\phi$ are real

(b) both $\sec^{-1}\theta$ and $\sec^{-1}\phi$ are real

(c) both $\tan^{-1}\theta$ and $\tan^{-1}\phi$ are real

(d) none of these

6. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then the value of $25\,(x+y+z)\;-\frac{216}{(x^3+y^3+z^3)}$ must be

(a) 1 (b) 2

(c) 3 (d) none of these

One or More Than One Option(s) Correct Type

7. If $\cos^{-1}x+(\sin^{-1}y)^2=\frac{p\pi^2}{4}$ and

$(\cos^{-1}x)(\sin^{-1}y)^2=\frac{\pi^2}{16}$, then

(a) $0\le p\le\frac{4}{\pi}+1$

(b) $p = 2$ is the integral value of p

(c) $p = 0, 1, 2$ (integral values)

(d) none of these

8. For $0<\phi<\frac{\pi}{2}$, if $x=\sum_{n=0}^{\infty}\cos^{2n}\phi$, $y=\sum_{n=0}^{\infty}\sin^{2n}\phi$, $z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$, then

(a) $xyz = xz + y$ (b) $xyz = xy + z$

(c) $xyz = x + y + z$ (d) $xyz = yz + x$

9. For the equation

$2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$,

which of the following is invalid?

(a) $a^2x + 2a = x$ (b) $a^2 + 2ax + 1 = 0$

(c) $a \ne 0$ (d) $a \ne -1, 1$

10. $\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right)+\tan^{-1}\left(\frac{a_2-a_1}{a_1a_2+1}\right)+\tan^{-1}\left(\frac{a_3-a_2}{a_2a_3+1}\right)$

$+.....+\tan^{-1}\left(\frac{a_n-a_{n-1}}{a_{n-1}a_n+1}\right)+\tan^{-1}\frac{1}{a_n}$ is equal to

(a) $\tan^{-1} xy$ (b) $\tan^{-1} \frac{x}{y}$

(c) $\tan^{-1} \frac{y}{x}$ (d) none of these

11. The value of θ for which

$$\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right), \text{ is/are:}$$

(a) $n\pi + \tan^{-1}(-2)$ (b) $n\pi,\ n\pi + \frac{\pi}{4}$

(c) $n\pi + \cot^{-1}(-2)$ (d) none of these

12. If x, y and z are in A.P. and $\tan^{-1} x$, $\tan^{-1} y$ and $\tan^{-1} z$ are also in A.P. then

(a) $x = y = z$ (b) $2x = 3y = 6z$

(c) $6x = 3y = 2z$ (d) $6x = 4y = 3z$

13. In ΔABC, $\angle C = \frac{\pi}{2}$ and

$$\sin^{-1} x = \sin^{-1}\left(\frac{ax}{c}\right) + \sin^{-1}\left(\frac{bx}{c}\right)$$

where a, b and c are the sides of triangle, then the values of x is/are

(a) 0 (b) 1 (c) 2 (d) –1

Comprehension Type

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{x_r - x_{r-1}}{1 + x_{r-1}x_r}\right) = \sum_{r=1}^{n}\left(\tan^{-1} x_r - \tan^{-1} x_{r-1}\right)$$

$$= \tan^{-1} x_n - \tan^{-1} x_0, \forall n \in N$$

On the basis of above information, answer the following questions:

14. The value of $\text{cosec}^{-1}\sqrt{5} + \text{cosec}^{-1}\sqrt{65} +$

$\text{cosec}^{-1}\sqrt{(325)} + \ldots$. to ∞ is

(a) π (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

15. The sum to infinite terms of the series

$$\tan^{-1}\left(\frac{2}{1-1^2+1^4}\right) + \tan^{-1}\left(\frac{4}{1-2^2+2^4}\right) + \tan^{-1}\left(\frac{6}{1-3^2+3^4}\right) + \ldots \text{ is}$$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) none of these

Matrix Match Type

16. Match the following :

Column I		Column II	
P.	If $2\tan^{-1}(2x+1) = \cos^{-1}(-x)$, then x is	1.	$-\frac{1}{\sqrt{2}}$
Q.	If $2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, then x is	2.	$\frac{\sqrt{3}}{2}$
R.	If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, then x is	3.	–1
		4.	0

	P	Q	R
(a)	2	1	3
(b)	3	2	1
(c)	3	1	2
(d)	4	2	1

Integer Answer Type

17. The number of solutions for the equation

$$2\sin^{-1}\sqrt{(x^2-x+1)} + \cos^{-1}\sqrt{(x^2-x)} = \frac{3\pi}{2} \text{ is}$$

18. If $\tan\theta + \tan\left(\frac{\pi}{3}+\theta\right) + \tan\left(-\frac{\pi}{3}+\theta\right) = a\tan 3\theta$, then a is equal to

19. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$ and $f(1) = 2$, $f(p+q) = f(p).f(q)$, $\forall\ p, q \in R$, then

$$x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{(x+y+z)}{x^{f(1)} + y^{f(2)} + z^{f(3)}} \text{ is equal to}$$

20. If $\lambda = \tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$, then the value of $-17\lambda^2$ must be

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted .....

No. of questions correct .....

Marks scored in percentage .....

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

BRAIN @ WORK

Limits, Continuity and Differentiability

LEFT AND RIGHT LIMITS

Let $y = f(x)$ be a given function, and $x = a$ is the point under consideration. Left tendency of $f(x)$ at $x = a$ is called its left limit and right tendency is called its right limit.

$f(a+0) = \lim_{h\to 0} f(a+h)$ and $f(a-0) = \lim_{h\to 0} f(a-h)$

where 'h' is a small positive number.

Thus for the existence of the limit of $f(x)$ at $x = a$, $f(a-0) = f(a+0)$

Reasons for non-existence of the limit

$\lim_{x\to a} f(x)$ will not exist due to any of these three reasons :

(a) $f(x)$ is not defined in the neighbourhood of $x = a$.

(b) Left and right limits of $f(x)$ are not the same.

SOME STANDARD LIMITS

- $\lim_{x\to 0} \frac{\sin ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\tan ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\sin^{-1} ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\tan^{-1} ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} (1+ax)^{1/x} = e^a = \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x, \forall \ a \in R$
- $\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e \ \ (a>0, a\neq 1)$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, \ \ a>0$
- $\lim_{x\to\infty} \frac{\log x}{x^m} = 0 \quad (m>0)$
- $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$, where m,n are rational numbers
- $\lim_{x\to 0} \frac{(1+x)^m - 1}{x} = m$

SOME THEOREMS ON LIMITS

- Let $f(x)$ and $g(x)$ are two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist finitely, then

(i) $\lim_{x\to a} (f \pm g)(x)$ and $\lim_{x\to a} f(x)\,g(x)$ exist.

(ii) If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists.

However, the converse of any of the above is not necessarily true.

- $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ provided $\lim_{x\to a} g(x)$ exists and lies in the domain of $f(x)$.

METHODS FOR CALCULATING THE LIMITS

1. When the limit is of the form $\lim_{x\to a}(f(x))^{g(x)}$ and $\lim_{x\to a} g(x) = \infty$

Here we discuss two different cases,

(i) When $\lim_{x\to a} f(x) = 1$.

In this case, $\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x)(f(x)-1)}$

(ii) When $\lim_{x\to a} f(x) \neq 1$, but $f(x)$ is positive in the neigbourhood of $x = a$.

In this case we write, $(f(x))^{g(x)} = e^{g(x)\ln f(x)}$

$\Rightarrow \lim_{x\to a} f(x))^{g(x)} = e^{\lim_{x\to a} g(x)\cdot \ln(f(x))}$

2. L'Hospital's Rule for Calculating Limits

Let f and g be two differentiable functions in the neighbourhood of a which satisfy

(i) $g'(x) \neq 0$ for any x in the neighbourhood of a

(ii) $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$.

Then, $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

This is L′ Hospital's Rule for $\dfrac{0}{0}$ form. However if $\lim\limits_{x\to a} f'(x) = 0$ and $\lim\limits_{x\to a} g'(x) = 0$, then again we can apply the L′ Hospital's rule.

3. Sandwich Theorem

Let f, g and h be three functions such that in the neighbourhood of a, $f(x) \le g(x) \le h(x)$ and $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$, then $\lim\limits_{x\to a} g(x) = l$.

CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a)$

i.e. L.H.L.=R.H.L.

= Value of the function at a or $\lim\limits_{x\to a} f(x) = f(a)$

Note : $f(x)$ will be discontinuous at $x = a$ in any of the following cases:

(i) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist but are not equal.

(ii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

(iv) At least one of the limits does not exist.

1. Properties of a Continuous Function

- A function $f(x)$, continuous on the interval $[a, b]$ possesses the following properties:

(i) $f(x)$ is bounded on $[a, b]$, *i.e.* there exist m and M such that $m \le f(x) \le M$ for all $x \in [a, b]$. Here m and M are minimum and maximum values of $f(x)$ respectively.

(ii) Let A be any number such that $\min f(x) \le A \le \max f(x)$, then there exists a point $x_0 \in [a, b]$ such that $f(x_0) = A$. It is called intermediate value theorem.

- If f and g are continuous at $x = a$, then $f + g$, kf and fg are continuous at $x = a$. Moreover, if $g(a) \neq 0$, then f/g is also continuous at $x = a$. Further if g is continuous at a and f is continuous at $g(a)$, then the composition fog is continuous at $x = a$.

DIFFERENTIABILITY

Let $y = f(x)$ be a continuous function at a point $x = a$.

It is said to be differentiable at $x = a$ if $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is finite.

If this limit exists, we call it the derivative of $y = f(x)$ at a and is denoted by $f'(a)$.

RIGHT AND LEFT HAND DERIVATIVES

- Right hand derivative of $f(x)$ at $x = a$ is given by

$$Rf'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

- Left hand derivative of $f(x)$ at $x = a$ is given by

$$Lf'(a) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$

Clearly, $f(x)$ is differentiable at $x = a$ if and only if $Rf'(a) = Lf'(a)$.

Note : (i) If a function $f(x)$ is differentiable at $x = a$ then, it is also continuous at $x = a$. But if a function is continuous at a point, it is not necessarily differentiable at that point.

(ii) Geometrically, we interpret $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ as the slope of the graph at the point $(a, f(a))$. The line through $(a, f(a))$ which has this slope is called the tangent line at $(a, f(a))$. Thus, if there is no tangent line at a certain point, the function is not differentiable at that point. In other words, a function is not differentiable at a point where the curve suddenly changes direction (corner point) or the tangent is vertical at some point.

PROBLEMS

1. Which one of the following is not always true?

(a) If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.

(b) If $f(x)$ is continuous at $x = a$, then it is differentiable at $x = a$.

(c) If $f(x)$ and $g(x)$ are differentiable at $x = a$, then $f(x) + g(x)$ is also differentiable at $x = a$.

(d) If a function $f(x)$ is continuous at $x = a$, then $\lim\limits_{x\to a} f(x)$ exists.

2. The function $f(x) = \begin{cases} 2x+1, & x \in Q \\ (x-1)^2 + 4, & x \notin Q \end{cases}$ is

(a) continuous nowhere

(b) differentiable nowhere

(c) continuous but not differentiable exactly at one point

(d) differentiable and continuous only at one point and discontinuous elsewhere

3. The value of $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\int_{\pi/2}^{x} t\,dt}{\sin(2x-\pi)}$ is

(a) ∞ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{8}$

4. The value of p for which the function

$$f(x)=\begin{cases}\dfrac{(4^x-1)^3}{\sin\frac{x}{p}\log\left[1+\frac{x^2}{3}\right]}, & x\neq 0\\ 12(\log 4)^3, & x=0\end{cases}$$

may be continuous at $x = 0$, is

(a) 1 (b) 2
(c) 3 (d) none of these

5. ABC is an isosceles triangle inscribed in a circle of radius r . If $AB = AC$ and h is the altitude from A to BC and p be the perimeter of ABC then $\lim\limits_{h\to 0}\dfrac{\Delta}{p^3}$ equals (where Δ is the area of the triangle)

(a) $\dfrac{1}{32r}$ (b) $\dfrac{1}{64r}$
(c) $\dfrac{1}{128r}$ (d) none of these

6. The function $f(x) = [x]^2 - [x^2]$, (where [.] is the greatest integer function), is discontinuous at

(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

7. If $f(x) = a|\sin x| + be^{|x|} + c|x|^3$ and if $f(x)$ is differentiable at $x = 0$, then

(a) $b = 0, c = 0, a$ is any real
(b) $a = 0, b = 0, c$ is any real
(c) $c = 0, a = 0, b$ is any real
(d) $a + b = 0$

8. Let $f : R \to R$ be a function defined by $f(x) = \min\{x + 1, |x| + 1\}$. Which of the following is true?

(a) $f(x) \geq 1$ for all $x \in R$.
(b) $f(x)$ is not differentiable at $x = 1$.
(c) $f(x)$ is differentiable everywhere.
(d) $f(x)$ is not differentiable at $x = 0$.

9. Let α and β be the roots of the equation , $ax^2 + bx + c = 0$ where $1 < \alpha < \beta$. If $\lim\limits_{x\to m}\dfrac{|ax^2+bx+c|}{ax^2+bx+c}=1$, then

(a) $a > 0$ and $m > 1$
(b) $a < 0$ and $m < 1$
(c) $a < 0$ and $\alpha < m < \beta$
(d) $\dfrac{|a|}{a}=1$ and $m > \alpha$

10. The function $f : R - \{0\} \to R$, $f(x)=\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining $f(0)$ as

(a) 2 (b) –1 (c) 0 (d) 1

11. If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$. Then the value of $f^2(4) + g^2(4)$ is

(a) 8 (b) 16 (c) 32 (d) 64

12. The number of values of a for the which the function $f(x) = (x + 1)|x - a|$ is differentiable $\forall\, x \in R$, is

(a) 0 (b) 1
(c) 2 (d) more than 2

13. If $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

(a) $a = 1, b = 1$ (b) $a = 1, b = 2$
(c) $a = 1, b = -2$ (d) none of these

14. $\lim\limits_{x\to\infty}\left(\dfrac{1}{e}-\dfrac{x}{1+x}\right)^x$ is equal to

(a) $\dfrac{e}{1-e}$ (b) 0
(c) $\dfrac{e}{e^{1-e}}$ (d) does not exist

15. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$ is equal to

(a) 0 (b) ∞
(c) $\dfrac{1}{2}$ (d) none of these

16. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?

I. f is continuous on the closed interval $[a, b]$.
II. f is bounded on the open interval (a, b).
III. If $a < a_1 < b_1 < b$ and $f(a_1) < 0 < f(b_1)$, then there is a number c such that $a_1 < c < b_1$ and $f(c)=0$.

(a) I and II only (b) I and III only
(c) II and III only (d) only III

17. $\lim_{x\to\infty} \frac{2\sqrt{x}+3\sqrt[3]{x}+4\sqrt[4]{x}+.....+n\sqrt[n]{x}}{\sqrt{(2x-3)}+\sqrt[3]{(2x-3)}+.....+\sqrt[n]{(2x-3)}}$ is equal to

(a) 1 (b) ∞
(c) $\sqrt{2}$ (d) none of these

18. If $f(x)=\lim_{n\to\infty}\left(x^{\frac{1}{n}}-1\right)n$, then for $x, y>0, f(xy)$ is equal to

(a) $f(x)\,f(y)$ (b) $f(x)+f(y)$
(c) $f(x)-f(y)$ (d) none of these

19. The value of $\lim_{n\to\infty}\left(\sum_{r=1}^{n}\left(\frac{2r+5}{r^2+r}\right)\left(\frac{3}{5}\right)^{r+1}\right)$ is equal to

(a) 0 (b) $\frac{9}{5}$
(c) $\frac{3}{5}$ (d) none of these

20. $\lim_{x\to0}\left\{(1+x)^{\frac{2}{x}}\right\}$ (where $\{\cdot\}$ denotes the fractional part of x) is equal to

(a) e^2-7 (b) e^2-8
(c) e^2-6 (d) none of these

21. $\lim_{x\to0}\frac{\cos(\tan x)-\cos x}{x^4}$ is equal to

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1

22. Among (i) $\lim_{x\to\infty}\sec^{-1}\left(\frac{x}{\sin x}\right)$ and (ii) $\lim_{x\to\infty}\sec^{-1}\left(\frac{\sin x}{x}\right)$

(a) (i) exists, (ii) does not exist
(b) (i) does not exist, (ii) exists
(c) both (i) and (ii) exist
(d) neither (i) nor (ii) exists

23. $\lim_{x\to1}\frac{(1-x)(1-x^2).....(1-x^{2n})}{\{(1-x)(1-x^2)....(1-x^n)\}^2}, n\in N=$

(a) ${}^{2n}P_n$ (b) ${}^{2n}C_n$
(c) $2n!$ (d) none of these

24. If $\lim_{x\to0}(x^{-3}\sin3x+ax^{-2}+b)$ exists and is equal to 0, then

(a) $a=-3$ and $b=\frac{9}{2}$ (b) $a=3$ and $b=\frac{9}{2}$
(c) $a=-3$ and $b=\frac{-9}{2}$ (d) $a=3$ and $b=\frac{-9}{2}$

25. If $f(x)=\begin{cases}3(1+|\tan x|)^{\frac{\alpha}{|\tan x|}}; \frac{-1}{2}<x<0\\ \beta \qquad\qquad ; x=0\\ 3\left(1+\left|\frac{\sin x}{3}\right|\right)^{\frac{6}{|\sin x|}}; 0<x<\frac{2}{3}\end{cases}$

is a continuous function at $x=0$, then the ordered pair (α, β) is equal to

(a) $(2, e^2)$ (b) $\left(2, \frac{2}{e^2}\right)$
(c) $(2, 3e^2)$ (d) $\left(2, \frac{3}{e^2}\right)$

26. For $x>0$, let $h(x)$

$=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q} \text{ where } p \text{ and } q>0 \text{ are relatively prime integers}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$

then which one of the following does not hold good?

(a) $h(x)$ is discontinuous for all x in $(0, \infty)$
(b) $h(x)$ is continuous for each irrational in $(0, \infty)$
(c) $h(x)$ is discontinuous for each rational in $(0, \infty)$
(d) $h(x)$ is not derivable for all x in $(0, \infty)$.

27. Let $f:(1, \infty)\to(0, \infty)$ be a continuous decreasing function with $\lim_{x\to\infty}\frac{f(4x)}{f(8x)}=1$. Then $\lim_{x\to\infty}\frac{f(6x)}{f(8x)}$ is equal to

(a) $\frac{4}{8}$ (b) $\frac{4}{6}$ (c) $\frac{6}{8}$ (d) 1

28. The value of $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m$ is

(a) 1 (b) e
(c) e^{-1} (d) none of these

29. Given $f(x)=\begin{cases}x \text{ for } |x|\le1\\ 1 \text{ for } |x|>1\end{cases}$,

$$g(x)=\begin{cases}\cos\left(\frac{\pi}{2}x\right) & \text{for } |x|\le 1\\ |x-1| & \text{for } |x|>1\end{cases},$$

$$h(x)=\begin{cases}\frac{(|x|-1)}{\log_a|x|} & \text{for } |x|\neq 1\\ \ln a & \text{for } |x|=1,\ a>0,\ a\neq 1\end{cases}$$

If l, m, n denote the number of points of discontinuity of the functions f, g and h in their domain respectively, over R, then (l, m, n) is

(a) $(0, 0, 0)$ (b) $(1, 1, 1)$

(c) $(2, 2, 2)$ (d) $(1, 1, 0)$

30. $\lim\limits_{x\to 0}\left(\frac{1^x+2^x+3^x+....+n^x}{n}\right)^{\frac{1}{x}}$ is equal to

(a) $(n!)^n$ (b) $(n!)^{1/n}$

(c) $n!$ (d) $\ln(n!)$

31. If $\lim\limits_{x\to 0^-}\sin^{-1}[\tan x]=l$, then value of $\{l\}$ is equal to (where [.] and {.} denotes greatest integer function and fractional part of a function)

(a) 0 (b) $1-\frac{\pi}{2}$ (c) $\frac{\pi}{2}-1$ (d) $2-\frac{\pi}{2}$

32. If $f(x) = |1 - x|$, then the number of points where $g(x) = \sin^{-1}(f(|x|))$ is non-differentiable, are

(a) 2 (b) 3 (c) 4 (d) 5

33. Let $f(x)=\frac{1}{1+e^{\frac{1}{x}}}$ for $x\neq 0$ and $f(0) = 0$, then

(a) $f(0^+)$ does not exist (b) $f(0^-) = 0$

(c) $f'(0^-) = 1$ (d) $f'(0^+) = 0$

34. Let $f(x)=\lim\limits_{n\to\infty}\frac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}$. Then the set of values of x for which $f(x) = 0$ is

(a) $|2x|>\sqrt{3}$ (b) $|2x|<\sqrt{3}$

(c) $|2x|\ge\sqrt{3}$ (d) $|2x|\le\sqrt{3}$

35. If $f(x)=\begin{cases}x^n\sin\left(\frac{1}{x^2}\right) & , x\neq 0, n\in I\\ 0 & , x=0\end{cases}$

Then,

(a) $\lim\limits_{x\to 0}f(x)$ exists for $n>1$

(b) $\lim\limits_{x\to 0}f(x)$ exists for $n<0$

(c) $\lim\limits_{x\to 0}f(x)$ does not exists for any value of n

(d) $\lim\limits_{x\to 0}f(x)$ cannot be determined

SOLUTIONS

1. (b) : If $f(x)$ is continuous at $x = a$, then it is not always differentiable at $x = a$.

2. (d) : $y = (x - 1)^2 + 4 = x^2 - 2x + 5$

According to question,

$y = x^2 - 2x + 5 = 2x + 1$

$\Rightarrow\quad x^2 - 4x + 4 = 0 = (x - 2)^2 = 0 \quad\Rightarrow\quad x = 2$

$\therefore$ $f(x)$ is continuous and differentiable at $x = 2$ only.

3. (c) : Let $y=\lim\limits_{x\to\pi/2}\frac{\int_{\pi/2}^{x}t\,dt}{\sin(2x-\pi)}\Rightarrow y=\lim\limits_{x\to\pi/2}\frac{\left[\frac{t^2}{2}\right]_{\pi/2}^{x}}{\sin(2x-\pi)}$

$$\therefore\ y=\lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{x^2}{2}-\frac{\pi^2}{8}\right)}{\sin(2x-\pi)}\Rightarrow y=\lim_{x\to\frac{\pi}{2}}\frac{\left(4x^2-\pi^2\right)}{8[\sin(2x-\pi)]}$$

$$\Rightarrow\quad y=\lim_{x\to\pi/2}\frac{1}{8}\frac{(2x-\pi)(2x+\pi)}{\sin(2x-\pi)}$$

$$\therefore\quad y=\frac{1}{8}\frac{\lim\limits_{x\to\pi/2}(2x+\pi)}{\lim\limits_{x\to\pi/2}\frac{\sin(2x-\pi)}{(2x-\pi)}}\quad\Rightarrow\quad y=\frac{1}{8}\times 2\pi=\frac{\pi}{4}$$

4. (d) : For $f(x)$ to be continuous at $x = 0$, we have

$$\lim_{x\to 0}f(x)=f(0)=12(\log 4)^3 \qquad ...(i)$$

Now, $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\left(\frac{4^x-1}{x}\right)^3\times\frac{\left(\frac{x}{p}\right)}{\left(\sin\frac{x}{p}\right)}\cdot\frac{px^2}{\log\left(1+\frac{1}{3}x^2\right)}$

$$=(\log 4)^3\cdot 1\cdot p\cdot\lim_{x\to 0}\left(\frac{x^2}{\frac{1}{3}x^2-\frac{1}{18}x^4+\ldots}\right)=3p\,(\log 4)^3$$

$\therefore$ On comparing with (i), we get $p = 4$.

5. (c) : We have, $AB=\sqrt{h^2+2rh-h^2}=\sqrt{2rh}$

$p=2\sqrt{2rh}+2\sqrt{2rh-h^2}=2[\sqrt{2rh}+\sqrt{2rh-h^2}]$

Now, $\Delta=\frac{1}{2}2h\sqrt{2rh-h^2}=h\sqrt{2rh-h^2}$

$$\therefore \lim_{h\to 0}\frac{\Delta}{p^3}$$

$$=\lim_{h\to 0}\frac{h\sqrt{2rh-h^2}}{8[\sqrt{2rh}+\sqrt{2rh-h^2}\,]^3}$$

$$\therefore \lim_{h\to 0}\frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}[\sqrt{2r}+\sqrt{2r-h}]^3}$$

$$=\frac{\sqrt{2r}}{8\cdot 8(2r)(2r)^{1/2}}=\frac{1}{128r}$$

6. (d) : Given $f(x) = [x]^2 - [x^2]$

Let $x = n,\ n \in I$

Now, $\lim\limits_{x\to n^+} f(x)=n^2-n^2=0$

and $\lim\limits_{x\to n^-} f(x)=(n-1)^2-(n^2-1)=-2n+2$

Also, $f(n) = 0$

For the function to be continuous at $x = n$

$-2n + 2 = 0 \Rightarrow\ n = 1$

Hence, function is discontinuous at all integers except 1.

7. (d) : $f'(0^-)=\lim\limits_{h\to 0}\dfrac{a\sin h+be^h+ch^3-b}{-h}=-(a+b)$

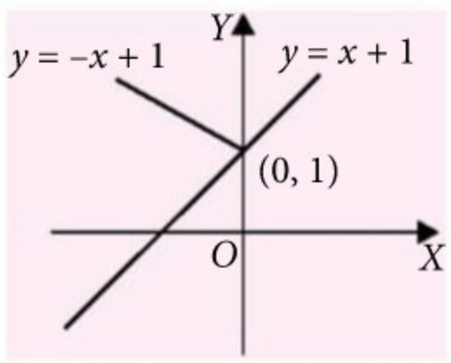
$$f'(0^+)=\lim_{h\to 0}\frac{a\sin h+be^h+ch^3-b}{h}$$

$$=\frac{a(\sin h)}{h}+\frac{b(e^h-1)}{h}+\frac{ch^3}{h}=a+b$$

$\because$ $f(x)$ is differentiable at $x = 0$

$\therefore a + b = -(a + b) \Rightarrow\ (a + b) = 0.$

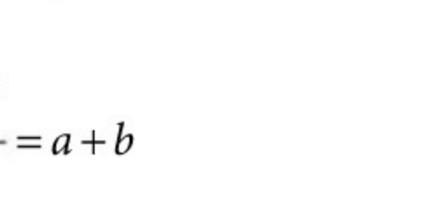
8. (c) : Given, $f(x) = \min\{x + 1, |x| + 1\}$



$\therefore$ $f(x) = x + 1\ \forall\ x \in R$

9. (c) : Given, $\lim\limits_{x\to m}\dfrac{|ax^2+bx+c|}{ax^2+bx+c}=1$

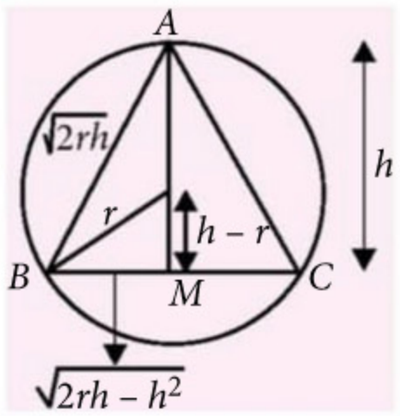
$$=\lim_{x\to m}\frac{|a(x-\alpha)(x-\beta)|}{a(x-\alpha)(x-\beta)}$$



If $a < 0$ and $\alpha < m < \beta$, then

$a\ (m - \alpha)\ (m - \beta) > 0$

$\Rightarrow$ Limit is 1.



10. (d) : $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}\right)=\lim\limits_{x\to 0}\dfrac{e^{2x}-1-2x}{x(e^{2x}-1)}$

$$=\lim_{x\to 0}\frac{\left\{\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots\ldots\right)-1-2x\right\}}{x\left\{\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots\ldots\right)-1\right\}}=1=f(0)$$

11. (c) : Given, $f'(x) = g(x)$

$\therefore$ $f''(x) = g'(x) = -f(x) \quad \Rightarrow f''(x) + f(x) = 0$

Solution of the differential equation is given by

$f(x) = c_1 \cos x + c_2 \sin x$ ...(i)

$\Rightarrow$ $f'(x) = -c_1 \sin x + c_2 \cos x = g(x)$ ...(ii)

Now, $f(2) = 4 \Rightarrow c_1 \cos 2 + c_2 \sin 2 = 4$ ...(iii)

$f'(2) = 4 \Rightarrow -c_1 \sin 2 + c_2 \cos 2 = 4$ ...(iv)

Squaring and adding, (iii) and (iv), we get

$c_1^2 + c_2^2 = 32$ ...(v)

Now, $f(4) = c_1 \cos 4 + c_2 \sin 4$ ...(vi)

$g(4) = -c_1 \sin 4 + c_2 \cos 4$ ...(vii)

Squaring and adding, (vi) and (vii), we get

$f^2(4)+g^2(4)=c_1^2+c_2^2=32$ ...(viii)

$\therefore$ $f^2(4) + g^2(4) = 32$

12. (b) : If function is differentiable in R, then it must be differentiable at $x_0 = a$.

$$f'(a^+)=\lim_{h\to 0}\frac{(a+h+1)|h|-0}{h}=a+1$$

$$f'(a^-)=\lim_{h\to 0}\frac{(a-h+1)|-h|}{-h}=-(a+1)$$

$\therefore$ $f'(a^+) = f'(a^-) \Rightarrow a = -1$

13. (c) : Given, $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$

$$\Rightarrow \lim_{x\to\infty}\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}=2$$

$\Rightarrow$ $1 - a = 0$ and $-b = 2 \Rightarrow\ a = 1, b = -2.$

14. (d) : $\lim\limits_{x\to\infty}\left(\dfrac{1}{e}-\dfrac{x}{1+x}\right)^x=\lim\limits_{x\to\infty}\left(\dfrac{\frac{1}{e}-\frac{1}{\frac{1}{x}+1}}{}\right)^x$

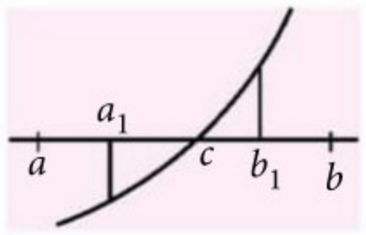
$$=\left(\frac{1}{e}-1\right)^{\infty}=(\text{some negative value})^{\infty}$$

which is not defined.

15. (c) : $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$

$$= \lim_{x\to\infty}\left(\frac{1+\frac{2}{x}-\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}}\right)^{\frac{2+1/x}{2-1/x}} = \frac{1}{2}$$

16. (d) : I and II are false. The function $f(x) = 1/x$, $0 < x < 1$, is a counter example.
Statement III is true.



Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$.

17. (c) : Since highest degree of x is $\frac{1}{2}$, divide numerator and denominator by $\sqrt{x}$, then value of limit becomes $\frac{2}{\sqrt{2}}$ or $\sqrt{2}$.

18. (b) : $\lim_{n\to\infty} n\left(x^{\frac{1}{n}}-1\right) = \lim_{n\to\infty}\frac{x^{1/n}-1}{\frac{1}{n}}$

$= \lim_{m\to 0}\frac{x^m-1}{m}\left(\text{where } \frac{1}{n} \text{ is replaced by } m\right) = \ln x$

Now, $f(xy) = \ln(xy) = \ln x + \ln y = f(x) + f(y)$.

19. (b) : Let $T_r = \left(\frac{2r+5}{r(r+1)}\right)\left(\frac{3}{5}\right)^{r+1}$

$= \left(\frac{5(r+1)-3r}{r(r+1)}\right)\left(\frac{3}{5}\right)^{r+1} = 3\left[\frac{1}{r}\left(\frac{3}{5}\right)^{r} - \frac{1}{r+1}\left(\frac{3}{5}\right)^{r+1}\right]$

$= 3(v_r - v_{r+1})$

So, $S_n = \sum_{r=1}^{n} T_r = 3\left[\sum_{r=1}^{n} v_r - \sum_{r=1}^{n} v_{r+1}\right]$

$\Rightarrow \quad S_n = 3(v_1 - v_{n+1}) = \frac{9(n+1)5^n - 3^{n+2}}{(n+1)\,5^{n+1}}$

So, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{9(n+1)5^n - 3^{n+2}}{(n+1)\,5^{n+1}} = \frac{9}{5}$.

20. (a) : $\{(1+x)^{2/x}\} = (1+x)^{2/x} - [(1+x)^{2/x}]$

Now, $\lim_{x\to 0}(1+x)^{2/x} = e^2$

$\therefore \quad \lim_{x\to 0}\{(1+x)^{2/x}\} = e^2 - [e^2] = e^2 - 7.$

21. (b) : Since, $\cos(\tan x) - \cos x$

$= 2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)$

$\therefore \quad \lim_{x\to 0}\frac{\cos(\tan x)-\cos x}{x^4}$

$= \lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4}$

$= \lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4\left(\frac{x+\tan x}{2}\right)\left(\frac{x-\tan x}{2}\right)}\left(\frac{x^2-\tan^2 x}{4}\right)$

$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\tan^2 x}{x^4}$

$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\ldots\right)^2}{x^4}$

$= \frac{1}{2}\lim_{x\to 0}\frac{1}{x^2}\left(1-\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots\ldots\right)^2\right) = \frac{1}{3}$

22. (a) : (i) $\lim_{x\to\infty}\sec^{-1}\left(\frac{x}{\sin x}\right) = \sec^{-1}\left(\frac{\infty}{\sin\infty}\right)$

$= \sec^{-1}\left(\frac{\infty}{\text{any value between } -1 \text{ to } 1}\right)$

$= \sec^{-1}(\pm\infty) = \frac{\pi}{2}$

(ii) $\lim_{x\to\infty}\sec^{-1}\left(\frac{\sin x}{x}\right) = \sec^{-1}\left(\frac{\sin\infty}{\infty}\right)$

$= \sec^{-1}\left(\frac{\text{any value between } -1 \text{ to } 1}{\infty}\right)$

$= \sec^{-1}0 =$ not defined

Hence (i) exists but (ii) does not exist.

23. (b) : $\lim_{x\to 1}\frac{(1-x)(1-x^2).....(1-x^{2n})}{\{(1-x)(1-x^2)....(1-x^n)\}^2}$

$$= \lim_{x\to 1} \frac{\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\ldots\left(\frac{1-x^{2n}}{1-x}\right)}{\left(\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\ldots\left(\frac{1-x^{n}}{1-x}\right)\right)^2}$$

$$= \frac{1\times 2\times 3\ldots..(2n)}{(1\times 2\times 3\times \ldots.\times n)^2} = \frac{(2n)!}{n!n!} = {}^{2n}C_n$$

24. (a) : We have,

$$\lim_{x\to 0} \frac{\sin 3x}{x^3} + \frac{a}{x^2} + b = \lim_{x\to 0} \frac{\sin 3x + ax + bx^3}{x^3}$$

$$= \lim_{x\to 0} \frac{3\frac{\sin 3x}{3x} + a + bx^2}{x^2}$$

For existence of limit, $3 + a = 0 \quad \Rightarrow \quad a = -3.$

Now, $\lim_{x\to 0} \frac{\sin 3x - 3x + bx^3}{x^3} = 0$

$$= 27 \lim_{t\to 0} \frac{\sin t - t}{t^3} + b = 0 \qquad (\text{Put } 3x = t)$$

$$= \frac{-27}{6} + b = 0 \quad \Rightarrow \quad b = \frac{9}{2}\cdot$$

25. (c) : We have,

$$f(x) = \begin{cases} 3(1+|\tan x|)^{\frac{\alpha}{|\tan x|}}; & \frac{-1}{2} < x < 0 \\ \beta \qquad\qquad\qquad\quad ; & x = 0 \\ 3\left(1+\left|\frac{\sin x}{3}\right|\right)^{\frac{6}{|\sin x|}}; & 0 < x < \frac{2}{3} \end{cases}$$

Now, R.H.L $= \lim_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+} 3\left(1+\frac{|\sin x|}{3}\right)^{\frac{6}{|\sin x|}} = 3e^2 \qquad \ldots(i)$$

Similarly, L.H.L $= \lim_{x\to 0^-} f(x) = 3e^{\alpha} \qquad \ldots(ii)$

$\because$ $f(x)$ is continuous at $x = 0$

$\therefore \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

$\Rightarrow \quad 3e^{\alpha} = 3e^2 = \beta.$

So, $\alpha = 2, \beta = 3e^2.$

26. (a) : Let $x = \frac{2}{3}$ which is rational

$$\Rightarrow \quad h\left(\frac{2}{3}\right) = \frac{1}{3}$$

$$\lim_{t\to 0} h\left(\frac{2}{3} + t\right) = 0 \quad \Rightarrow \text{ Discontinuous at } x \in Q$$

Let $x = \sqrt{2} \notin Q \Rightarrow h(\sqrt{2}) = 0$

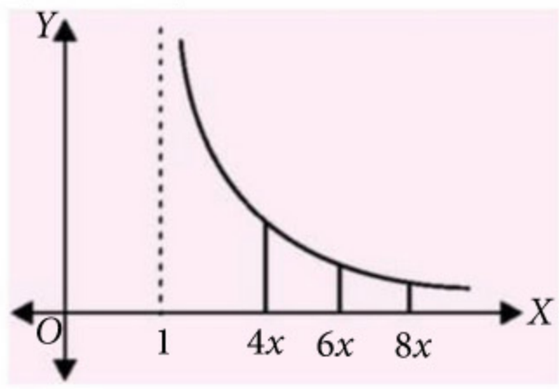
As $\sqrt{2} = 1.414213562$

$\therefore \quad \lim_{x\to \sqrt{2}^+} h(x) = \lim_{x\to \sqrt{2}^-} h(x) = h(\sqrt{2}) = 0$

Hence, $h(x)$ is continuous for all irrationals.

27. (d) : As $f(x)$ is a continuous decreasing positive function on $(1, \infty)$, so $0 < f(8x) < f(6x) < f(4x)$

$$\Rightarrow \quad 0 < 1 < \frac{f(6x)}{f(8x)} < \frac{f(4x)}{f(8x)}$$



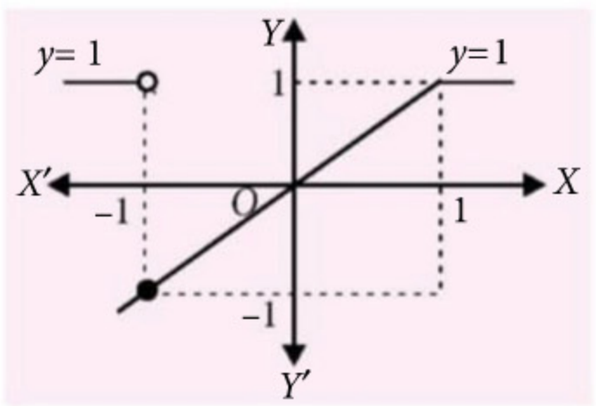
$$\Rightarrow \quad \lim_{x\to\infty} 1 \le \lim_{x\to\infty} \frac{f(6x)}{f(8x)} \le \lim_{x\to\infty} \frac{f(4x)}{f(8x)}$$

$\therefore$ By squeeze theorem, $\lim_{x\to\infty} \frac{f(6x)}{f(8x)} = 1$

28. (a) : $\lim_{m\to\infty} \left(\cos\frac{x}{m}\right)^m = \lim_{m\to\infty} \left[1-\left(1-\cos\frac{x}{m}\right)\right]^m$

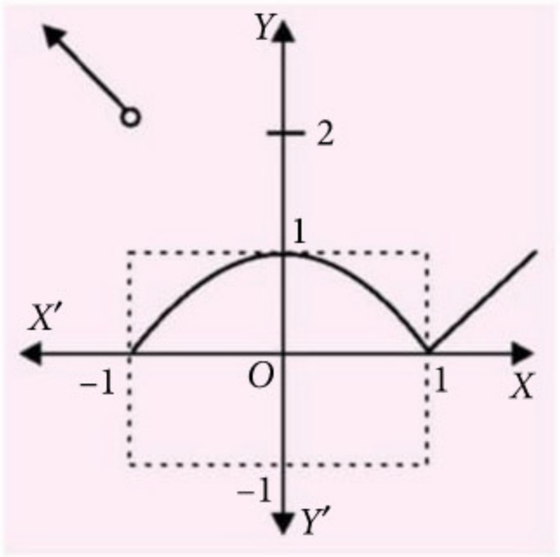
$$= \lim_{m\to\infty} \left[1-2\sin^2\frac{x}{2m}\right]^m = e^{\lim_{m\to\infty}\left(1-2\sin^2\frac{x}{2m}\right)m} = 1.$$

29. (d) : $f(x) = \begin{cases} x \text{ for } -1 \le x \le 1 \\ 1 \text{ for } x > 1 \text{ or } x < -1 \end{cases}$



Clearly, $f(x)$ is discontinuous at $x = -1$

$$g(x) = \begin{cases} \cos\frac{\pi x}{2} & \text{for } -1 \le x \le 1 \\ x-1 & \text{for } x > 1 \\ 1-x & \text{for } x < -1 \end{cases}$$



Clearly $g(x)$ is discontinuous at $x = -1$.

$$\text{Now, } h(x)=\begin{cases}\dfrac{x-1}{\log_a x}, & x>0,\ x\neq 1\\ \dfrac{-x-1}{\log_a(-x)}, & x<0,\ x\neq 1\\ \ln a, & |x|=1,\ a\neq 1\end{cases}$$

Clearly, $h(x)$ is continuous for all x including $x = 1$ or -1.

$\because$ $x = 0$ is not in the domain

$\therefore$ There is no point of discontinuity.

30. (b) : We have, $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+3^x+....+n^x}{n}\right)^{\frac{1}{x}}$

$$=e^{\lim\limits_{x\to 0}\left(\frac{1^x-1}{n}+\frac{2^x-1}{n}+....+\frac{n^x-1}{n}\right)\frac{1}{x}}$$

$$=e^{\lim\limits_{x\to 0}\frac{1}{n}\left\{\frac{1^x-1}{x}+\frac{2^x-1}{x}+....+\frac{n^x-1}{x}\right\}}$$

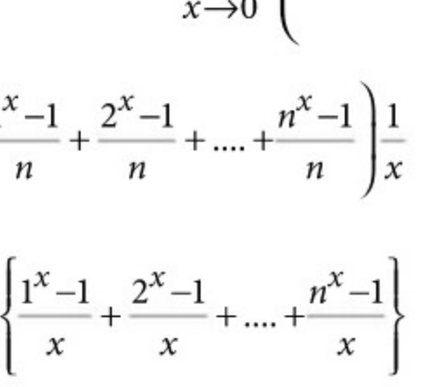
$$=e^{\frac{1}{n}[\log 1+\log 2+....+\log n]}=e^{\frac{1}{n}(\log n!)}=e^{(\log n!)^{1/n}}=(n!)^{1/n}$$

31. (d) : $\lim\limits_{x\to 0^-}\sin^{-1}[\tan x]=-\dfrac{\pi}{2}$;

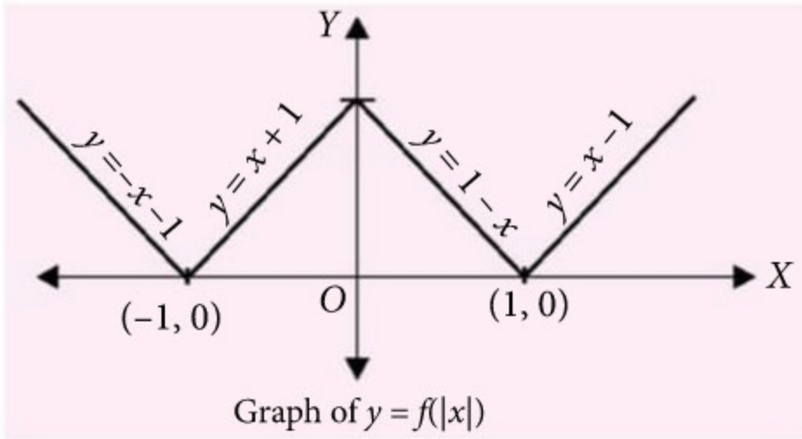
Hence, $\left\{-\dfrac{\pi}{2}\right\}=\left\{2-\dfrac{\pi}{2}-2\right\}=2-\dfrac{\pi}{2}$

where $[\cdot]$ and $\{\cdot\}$ denotes greatest integer and fractional part function.

32. (d) :



Graph of $y = f(x) = |x - 1|$



Graph of $y = f(|x|)$

Clearly, the domain of $g(x) = \sin^{-1}(f(|x|))$ is $[-2, 2]$

$\Rightarrow$ $g(x)$ is non-differentiable at $x = -2, -1, 0, 1, 2$.

$\therefore$ Number of non-derivable points are 5.

33. (d) : $f(x)=\begin{cases}\dfrac{1}{1+e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$

$$\therefore\quad f'(0^+)=\lim_{h\to 0}\frac{\frac{1}{1+e^{1/h}}}{h}=\lim_{h\to 0}\frac{1/h}{1+e^{1/h}}$$

$$=\lim_{h\to 0}\frac{-1/h^2}{e^{1/h}(-1/h^2)}=0$$

$$\text{Now, } f'(0^-)=\lim_{h\to 0}\frac{\frac{1}{1+e^{-1/h}}}{-h}\to -\infty;$$

$$\text{Also, } f(0^+)=\lim_{h\to 0}\frac{1}{1+e^{1/h}}=0;\ f(0^-)=\lim_{h\to 0}\frac{1}{1+e^{-1/h}}=1$$

34. (a) : Given $f(x)=\lim\limits_{n\to\infty}\dfrac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}=0$

$$\Rightarrow \lim_{n\to\infty}\left[\left(\frac{3}{\pi}\tan^{-1}2x\right)^2\right]^n\to\infty\ \Rightarrow\ \left(\frac{3}{\pi}\tan^{-1}2x\right)^2>1$$

$$\Rightarrow\ \left|\tan^{-1}2x\right|>\frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}2x<-\frac{\pi}{3}\text{ or }\tan^{-1}2x>\frac{\pi}{3}$$

$$\Rightarrow\quad 2x<-\sqrt{3}\text{ or }2x>\sqrt{3}\ \Rightarrow\ |2x|>\sqrt{3}$$

35. (a) : For $n > 1$,

$$\lim_{x\to 0}x^n\sin\left(\frac{1}{x^2}\right)=0\times(\text{any value between }-1\text{ to }1)=0$$

For $n < 0$,

$$\lim_{x\to 0}x^n\sin\left(\frac{1}{x^2}\right)=\infty\times(\text{any value between }-1\text{ to }1)=\infty$$

Challenging PROBLEMS for Entrance Exams

1. The value of $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ is

(a) 1 (b) $\frac{\sqrt{7}}{2}$ (c) $\frac{3\sqrt{3}}{4}$ (d) $\frac{\sqrt{15}}{4}$

2. The equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ can be solved if

(a) $-\sqrt{3} \le a \le \sqrt{3}$ (b) $-\sqrt{2} \le a \le \sqrt{2}$

(c) $-1 \le a \le 1$ (d) None of these

3. Given ΔABC is inscribed in the semicircle with diameter AB. The area of ΔABC equals 2/9 of the area of the semicircle. If the measure of the smallest angle in ΔABC is x, then $\sin 2x$ is equal to

(a) $\frac{\pi}{9}$ (b) $\frac{2\pi}{9}$ (c) $\frac{\pi}{18}$ (d) $\frac{\pi}{8}$

4. Set of all the values of x satisfying the inequality $\log_{x+\frac{1}{x}}\left(\log_2\frac{x-1}{x+2}\right)>0$ is

(a) $(-5, -2)$ (b) $(2, 5)$

(c) $(5, \infty)$ (d) ϕ

5. The integer value of x for which $x^2 + 19x + 92$ is square of an integer are

(a) -8 and -11 (b) -8 and 11

(c) 8 and 11 (d) None of these

6. Let a_n be the n^{th} term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \ne \beta$, then the common ratio is

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

7. The coefficient of $x^n y^n$ in the expansion of $\{(1 + x)(1 + y)(x + y)\}^n$ is

(a) $\sum_{r=0}^{n} C_r^2$ (b) $\sum_{r=0}^{n} C_r^3$

(c) $\sum_{r+s=0}^{n} {}^nC_r\,{}^nC_s^2$ (d) None of these

8. Given 6 different toys of red colour, 5 different toys of blue colour and 4 different toys of green colour. Combination of toys that can be chosen taking at least one red and one blue toys are

(a) 31258 (b) 31248

(c) 31268 (d) None of these

9. If length of common chord of two circles $x^2 + y^2 + 8x + 1 = 0$ and $x^2 + y^2 + 2\mu y - 1 = 0$ is $2\sqrt{6}$, then value of μ are

(a) ± 2 (b) ± 3

(c) ± 4 (d) None of these

10. If two distinct chords drawn from the point (4, 4) on the parabola $y^2 = 4ax$ are bisected on the line $y = mx$, then the set of value of m is given by

(a) $\left(\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right)$ (b) R

(c) $(0, \infty)$ (d) $(-2, 2)$

11. If the focal distance of an end of the minor axis of any ellipse (its axes as x and y axis respectively) is k and the distance between the foci is $2h$, then its equation is

(a) $\frac{x^2}{k^2}+\frac{y^2}{h^2}=1$ (b) $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$

(c) $\frac{x^2}{k^2}-\frac{y^2}{k^2-h^2}=1$ (d) $\frac{x^2}{k^2}+\frac{y^2}{k^2+h^2}=1$

12. If $A=\left\{x:\frac{\pi}{6}\le x\le\frac{\pi}{3}\right\}$ and $f(x) = \cos x - x(1 + x)$, then $f(A)$ is equal to

(a) $\left[\frac{1}{2}-\frac{\pi}{3}-\frac{\pi^2}{9}, \frac{\sqrt{3}}{2}-\frac{\pi}{6}-\frac{\pi^2}{36}\right]$

(b) $\left[\frac{1}{2}+\frac{\pi}{3}-\frac{\pi^2}{9},\frac{\sqrt{3}}{2}+\frac{\pi}{6}-\frac{\pi^2}{36}\right]$

(c) $\left(\frac{1}{2}-\frac{\pi}{3}-\frac{\pi^2}{9},\frac{\sqrt{3}}{2}-\frac{\pi}{6}-\frac{\pi^2}{36}\right)$

(d) None of these

13. If the function $f(x)=\left[\frac{(x-2)^3}{a}\right]\sin(x-2)+a\cos(x-2)$, (where $[\cdot]$ denotes the greatest integer function) is continuous and differentiable in $(4, 6)$, then

(a) $a\in[8, 64]$ (b) $a\in(0, 8]$

(c) $a\in[64, \infty)$ (d) None of these

14. If $f(x)=\begin{cases}1/|x|; & |x|\geq 1\\ ax^2+b; & -1<x<1\end{cases}$ is differentiable $\forall\, x$, then value of a and b is

(a) $a = 1/2, b = -3/2$ (b) $a = -1/2, b = 3/2$

(c) $a = 3/2, b = 1/2$ (d) None of these

15. If $f''(x) > 0$ and $f'(1) = 0$ such that $g(x) = f(\cot^2 x + 2\cot x + 2)$, where $0 < x < \pi$ then the interval in which $g(x)$ is decreasing is

(a) $(0, \pi)$ (b) $\left(\frac{\pi}{2},\pi\right)$

(c) $\left(\frac{3\pi}{4},\pi\right)$ (d) $\left(0,\frac{3\pi}{4}\right)$

16. If $I=\int\frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx$, then I equals

(a) $\log|x|+\log\left|1+\sqrt{1-x^2}\right|+\sin^{-1}\sqrt{x}+C$

(b) $\log|x|-\log\left|1-\sqrt{1-x^2}\right|+\tan^{-1}x+C$

(c) $\log|x|-\log\left|1+\sqrt{1-x^2}\right|-\sin^{-1}x+C$

(d) $\log\left|1+\sqrt{1-x^2}\right|-\log|x|+\cos^{-1}x+C$

17. If $f(x)=\tan^{-1}x+\ln\sqrt{1+x}-\ln\sqrt{1-x}$. The integral of $(1/2)f'(x)$ with respect to x^4 is

(a) $e^{-x^4}+c$ (b) $-\ln(1-x^4)+c$

(c) $e^{\sqrt{1-x^4}}+c$ (d) $\ln(1+x^2)+c$

18. The value of $\int_1^{16}\tan^{-1}\sqrt{\sqrt{x}-1}\,dx$ is

(a) $\frac{16\pi}{3}+2\sqrt{3}$ (b) $\frac{4}{3}\pi-2\sqrt{3}$

(c) $\frac{4}{3}\pi+2\sqrt{3}$ (d) $\frac{16}{3}\pi-2\sqrt{3}$

19. If f is a continuous function such that $\int_0^x f(t)dt\to\infty$ as $|x|\to\infty$, then for all $k\in R$, then $k^2x^2+\int_0^x f(t)dt-a=0(a>0)$ has

(a) all roots in $(-\infty, 0)$

(b) all roots in $(0, \infty)$

(c) odd number of roots in $(-\infty, 0)$ and odd number of roots in $(0, \infty)$

(d) None of these

20. Through any point (x, y) of a curve which passes through the origin, lines are drawn parallel to the coordinate axes. The curve, given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is twice the other, represents a family of

(a) circles (b) parabolas

(c) hyperbolas (d) straight lines

21. The area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$, the axis of abscissa and two ordinates corresponding to the points of minimum of the function $y(x)$ (in sq. units) is

(a) 10/3 (b) 27/10

(c) 21/10 (d) None of these

22. Solution of the equation $y=x\frac{dy}{dx}+\frac{dx}{dy}$ represents

(a) Family of straight lines and a parabola

(b) Family of straight lines and a hyperbola

(c) Family of circles and parabola

(d) None of these

23. The differential equation of family of parabola with foci at the origin and axis along the x axes is

(a) $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0$

(b) $x\left(\frac{dy}{dx}\right)^2+2y\frac{dy}{dx}-y=0$

(c) $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}+y=0$

(d) None of these

24. If z_1, z_2 are two complex numbers such that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$ and $iz_1 = Kz_2$, where $K\in R$, then the angle between $z_1 - z_2$ and $z_1 + z_2$ is

(a) $\tan^{-1}\left(\frac{2K}{K^2+1}\right)$ (b) $\tan^{-1}\left(\frac{K}{1-K^2}\right)$

(c) $-2\tan^{-1}K$ (d) $2\tan^{-1}K$

25. If α is a complex n^{th} root of unity and if Z_1 and Z_2 are two complex numbers, then $\sum_{r=0}^{n-1}|Z_1+\alpha^r Z_2|^2 =$

(a) $n^2|Z_1 + Z_2|^2$ (b) $\left(\frac{Z_1}{n}+\frac{Z_2}{n}\right)^2$

(c) $n(|Z_1|^2 + |Z_2|^2)$ (d) $n^2(|Z_1|^2 + |Z_2|^2)$

26. Two persons A and B throw a die alternately till one of them get a "six" and wins the game. The probability of winning of B if A starts first is

(a) $\frac{6}{11}$ (b) $\frac{5}{11}$

(c) $\frac{3}{11}$ (d) None of these

27. A person throws a dice while he gets a number greater than 2. The probability that he gets a 6 in the last thrown is

(a) 2/3 (b) 1/4 (c) 1/3 (d) 1/12

28. The values of λ for which the system of equations $x + y - 3 = 0$, $(1 + \lambda)x + (2 + \lambda)y - 8 = 0$, $x - (1 + \lambda)y + (2 + \lambda) = 0$ is consistent are

(a) $-5/3, 1$ (b) $2/3, -3$

(c) $-1/3, -3$ (d) $0, 0$

29. The number of values of k for which the system of equations $(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has no solution is

(a) 0 (b) 1 (c) 2 (d) infinite

30. If $A=\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}$ and $f(x) = 1 + x + x^2 + ... + x^{16}$, then $f(A)$ is equal to

(a) 0 (b) $\begin{bmatrix}1 & 5\\ 0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 5\\ 0 & 0\end{bmatrix}$ (d) $\begin{bmatrix}0 & 5\\ 1 & 1\end{bmatrix}$

31. If $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$ and $\vec{a}\perp(\vec{b}+\vec{c}), \vec{b}\perp(\vec{c}+\vec{a}), \vec{c}\perp(\vec{a}+\vec{b})$ then $|\vec{a}+\vec{b}+\vec{c}|$ is equal to

(a) 50 (b) 25

(c) $5\sqrt{2}$ (d) 12

32. If lines $\frac{x-1}{2}=\frac{y-2}{x_1}=\frac{z-3}{x_2}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ lies in the same plane then for equation $x_1t^2 + (x_2 + 2)t + a = 0$, which of the options holds?

(a) $2x_1 - x_2 = 1$

(b) Sum of roots of given equation = -2

(c) $2x_1 + x_2 = -4$

(d) Sum of roots of given equation = 0

33. The variance of the first n natural numbers is

(a) $\frac{n^2-1}{12}$ (b) $\frac{n^2-1}{6}$

(c) $\frac{n^2+1}{6}$ (d) $\frac{n^2+1}{12}$

SOLUTIONS

1. (b): Let $S=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ and

$C=\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}$

Now, $C + iS = \alpha + \alpha^2 + \alpha^4$ ...(i)

where, $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$, is complex 7^{th} root of unity

Also, $C - iS = \alpha^6 + \alpha^5 + \alpha^3$ ...(ii)

$\therefore\ \alpha^6=\bar{\alpha}, \alpha^5=\bar{\alpha}^2, \alpha^3=\bar{\alpha}^4$

Adding eqn. (i) and (ii), we get

$2C=\alpha+\alpha^2+\alpha^3+...+\alpha^6=\frac{\alpha^7-\alpha}{\alpha-1}=-1 \quad (\because\ \alpha^7=1)$

$\Rightarrow\ C=-1/2$

Multiplying eqn. (i) and (ii), we get

$C^2+S^2=2 \Rightarrow S=\frac{\sqrt{7}}{2}$

2. (b): We have, $\sin^4 x - 2\cos^2 x + a^2 = 0$

or, $y^2 - 2(1-y) + a^2 = 0$ [where $\sin^2 x = y$]

$\Rightarrow \; y^2 + 2y + a^2 - 2 = 0$

For y to be real, discriminant ≥ 0

$\Rightarrow \; 4 - 4(a^2 - 2) \geq 0$

$\Rightarrow \; a^2 \leq 3$ ...(i)

Since, $\sin^2 x = y \Rightarrow \; 0 \leq y \leq 1$

$\Rightarrow \quad 0 \leq \sqrt{3-a^2} - 1 \leq 1$

$\Rightarrow \; 1 \leq \sqrt{3-a^2} \leq 2 \Rightarrow 1 \leq 3 - a^2 \leq 4$

$\Rightarrow \; 2 - a^2 \geq 0 \Rightarrow a^2 \leq 2$

$\Rightarrow \; -\sqrt{2} \leq a \leq \sqrt{2}.$

3. (a): Since, $a^2 + b^2 = c^2 = 4r^2$ ...(i)

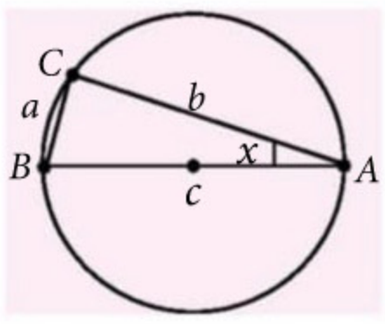
Also, $\frac{1}{2} a \cdot b = \frac{2}{9} \cdot \left(\frac{1}{2}\pi r^2\right)$

$\Rightarrow \; 9ab = 2\pi r^2$ ...(ii)

From eqs. (i) and (ii), we get

$$\frac{a^2+b^2}{9ab} = \frac{2}{\pi}$$

$$\Rightarrow \frac{a}{b} + \frac{b}{a} = \frac{18}{\pi}$$



Now, $\angle BAC = x$ then $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{18}{\pi}$

$$\Rightarrow \quad \frac{\cos^2 x + \sin^2 x}{\sin x \cdot \cos x} = \frac{18}{\pi}$$

$$\Rightarrow \quad \sin x \cdot \cos x = \frac{\pi}{18} \Rightarrow \sin 2x = \frac{\pi}{9}$$

4. (d): Since $\log_{x+\frac{1}{x}} \log_2 \left[\frac{x-1}{x+2}\right] > 0$

$\therefore \; \frac{x-1}{x+2} > 0$

$\Rightarrow \; x > 1, x < -2$ ...(i)

But $x + \frac{1}{x} > 0$

$\Rightarrow \; \frac{x^2+1}{x} > 0 \Rightarrow x > 0$ ...(ii)

From (i) and (ii), we get

$$x > 1; \log_{x+\frac{1}{x}} \left(\log_2 \frac{x-1}{x+2}\right) > 0$$

Take antilog both sides,

$$\log_2 \frac{x-1}{x+2} > 1$$

Take again antilog both sides,

$$\frac{x-1}{x+2} > 2 \Rightarrow \frac{x+5}{x+2} < 0$$

$\Rightarrow \; x \in (-5, -2)$ [But not true as $x > 1$]

5. (a): Let $x^2 + 19x + 92 = m^2$, $m \in I$

$x^2 + 19x + 92 - m^2 = 0$

$$\therefore \quad x = \frac{-19 \pm \sqrt{4m^2 - 7}}{2}$$

Since, $x \in I$, so $-19 \pm \sqrt{4m^2-7}$ must be an even integer

$\therefore \; \sqrt{4m^2-7}$ must be an odd integer

Let $\sqrt{4m^2-7} = 2k+1; k \in I$

$\therefore \; 4m^2 - (2k+1)^2 = 7$

$(2m - 2k - 1)(2m + 2k + 1) = 7$

Since, 7 is prime, so possible cases are as follows:

(i) $2m - 2k - 1 = 1$ and $2m + 2k + 1 = 7$ then $m = 2$

(ii) $2m - 2k - 1 = 7$ and $2m + 2k + 1 = 1$ then $m = 2$

$\therefore \;$ At $m = 2, x = -8, -11$

(iii) $2m - 2k - 1 = -1$ and $2m + 2k + 1 = -7$ then $m = -2$

(iv) $2m - 2k - 1 = -7$ and $2m + 2k + 1 = -1$ then $m = -2$

$\therefore \;$ At $m = -2, x = -8, -11$

6. (a): Let a G.P. with common ratio = r

Now, $\sum_{n=1}^{100} a_{2n} = a_2 + a_4 + a_6 + ... + a_{200}$

$= ar + ar^3 + ar^5 + ... + ar^{199} = \alpha$ ...(i)

Also, $\sum_{n=1}^{100} a_{2n-1} = a_1 + a_3 + a_5 + ... + a_{199}$

$= a + ar^2 + ar^4 + ... + ar^{198} = \beta$ ...(ii)

On dividing eqn. (ii) by eqn. (i), we get

$$\frac{r(a + ar^2 + ar^4 + ... + ar^{198})}{(a + ar^2 + ar^4 + ...ar^{198})} = \frac{\alpha}{\beta}$$

$$\Rightarrow \quad r = \frac{\alpha}{\beta}$$

7. (b): We have, $\{(1 + x)(1 + y)(x + y)\}^n$

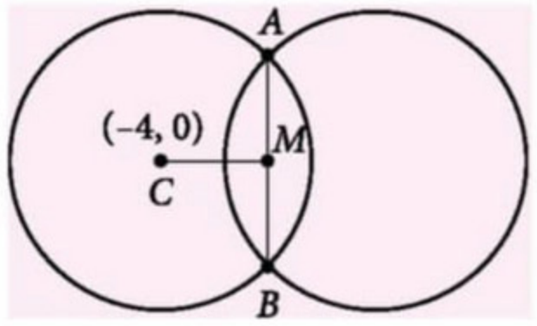
$= (C_0 + C_1 x + C_2 x^2 + ... + C_n x^n) \times (C_0 y^n + C_1 y^{n-1} + ... + C_{n-1} y + C_n) \times (C_0 x^n + C_1 x^{n-1} y + ... + C_{n-1} x y^{n-1} + C_n y^n)$

Coefficient of $x^n y^n$

$$= C_0^3 + C_1^3 + C_2^3 + ... + C_n^3 = \sum_{r=0}^{n} C_r^3$$

8. (b): At least one red toy can be chosen in
$= {}^6C_1 + {}^6C_2 + ... + {}^6C_6 = 2^6 - 1 = 63$ ways
At least one blue toy can be chosen in
$= {}^5C_1 + {}^5C_2 + ... + {}^5C_5 = 2^5 - 1 = 31$ ways
Green toys (with no restriction) can be selected in
$= {}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 = 16$ ways
$\therefore$ Total ways of selection $= 63 \times 31 \times 16 = 31248$

9. (b): Common chord is $S_1 - S_2 = 0$
$\Rightarrow 4x - \mu y + 1 = 0$



$\therefore \quad AC = \sqrt{15}$

$AM = \dfrac{AB}{2} = \sqrt{6}$

$\therefore \quad CM = 3$

$\therefore$ Perpendicular distance $= \dfrac{|-16+1|}{\sqrt{16+\mu^2}} = 3$

$\Rightarrow \mu = \pm 3$

10. (a): Any point on the line $y = mx$ can be taken as (t, mt).
Equation of the chord of parabola with (t, mt) as mid point
$y\,mt - 2(x + t) = m^2 t^2 - 4t$
Since, it passes through (4, 4),
$\therefore \quad 4mt - 2(4 + t) = m^2t^2 - 4t$
$\Rightarrow m^2t^2 - 2(2m + 1)t + 8 = 0$
For two such chords $D > 0$
$\Rightarrow (2m + 1)^2 - 8m^2 > 0$

$\Rightarrow \quad 4m^2 - 4m - 1 < 0 \Rightarrow \dfrac{1-\sqrt{2}}{2} < m < \dfrac{1+\sqrt{2}}{2}$

11. (b): Let equation of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and e is eccentricity of ellipse.
$\therefore \quad 2h = 2ae \Rightarrow ae = h$ ...(i)
Focal distance of one end of minor axis say $(0, b)$ is k,
$\therefore \quad a + e(0) = k \Rightarrow a = k$ ...(ii)
From (i) and (ii),
$b^2 = a^2(1 - e^2)$
$\Rightarrow b^2 = a^2 - a^2e^2 = k^2 - h^2$

$\therefore$ Equation of ellipse is $\dfrac{x^2}{k^2} + \dfrac{y^2}{k^2 - h^2} = 1.$

12. (a): We have $f(x) = \cos x - x(1 + x)$

Since, in the interval $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$, $\cos x$ decreases and $x(1 + x)$ increases.

$\therefore \quad f(x)$ decreases in $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$.

$\therefore \quad f\left(\dfrac{\pi}{3}\right) \le f(x) \le f\left(\dfrac{\pi}{6}\right); x \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

Hence, $f(A) = \left[\dfrac{1}{2} - \dfrac{\pi}{3}\left(1 + \dfrac{\pi}{3}\right), \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}\left(1 + \dfrac{\pi}{6}\right)\right]$

$= \left[\dfrac{1}{2} - \dfrac{\pi}{3} - \dfrac{\pi^2}{9}, \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6} - \dfrac{\pi^2}{36}\right]$

13. (c): We have, $x \in (4, 6)$
$\Rightarrow 2 < x - 2 < 4$

$\Rightarrow \quad \dfrac{8}{a} < \dfrac{(x-2)^3}{a} < \dfrac{64}{a}, \quad a > 0$

For $f(x)$ to be continuous and differentiable in (4, 6), $\left[\dfrac{(x-2)^3}{a}\right]$ must attain a constant value for $x \in (4, 6)$.

Clearly, this is possible only when $a \ge 64$
In that case, we have
$f(x) = a\cos(x - 2)$ which is continuous and differentiable.
$\therefore \quad a \in [64, \infty)$

14. (b): $f(x) = \begin{cases} 1/|x|; & |x| \ge 1 \\ ax^2 + b; & -1 < x < 1 \end{cases}$

$f(x) = \begin{cases} -\dfrac{1}{x}; & x \le -1 \\ ax^2 + b; & -1 < x < 1 \\ \dfrac{1}{x}; & x \ge 1 \end{cases}$

Since $f(x)$ is continuous at $x = 1$.
$\therefore \quad a \cdot 1 + b = 1$ ...(i)
Also, $f(x)$ is differentiable at $x = 1$.

$\therefore \quad 2ax = -\dfrac{1}{x^2}$

At $x = 1$, $2a = -1$
$\Rightarrow \quad a = -1/2$
Using (i), we get, $b = 3/2$

15. (d): Here, $g(x) = f(\cot^2 x + 2\cot x + 2)$
$\Rightarrow g'(x) = f'(\cot^2 x + 2\cot x + 2)$
$\{-2\cot x \operatorname{cosec}^2 x - 2\operatorname{cosec}^2 x\}$
For $g(x)$ to be decreasing, $g'(x) < 0$
$\Rightarrow f'\{(\cot x + 1)^2 + 1\} \cdot (-2\operatorname{cosec}^2 x)(\cot x + 1) < 0$
$\Rightarrow f'\{(\cot x + 1)^2 + 1\} \cdot (\cot x + 1) > 0$ ...(i)
{as $f''(x) > 0 \Rightarrow f'(x)$ is increasing, then

$f'\left\{(\cot x + 1)^2 + 1 > f'(1) = 0 \; \forall x \in \left(0, \dfrac{3\pi}{4}\right) \cup \left(\dfrac{3\pi}{4}, \pi\right)\right\}$

Thus, equation (i) holds, if $\cot x + 1 > 0$

$\Rightarrow \quad \cot x > -1 \; \forall x \in \left(0, \frac{3\pi}{4}\right)$

16. (c) : $I = \int \left(\frac{1}{x}\right) \frac{1-x}{\sqrt{1-x^2}} dx = \int \frac{dx}{x\sqrt{1-x^2}} - \int \frac{dx}{\sqrt{1-x^2}}$

In the first integral, put $x = \frac{1}{t}$, we get

$$I = \int \frac{(-1/t^2)\,dt}{\frac{1}{t}\sqrt{1-\frac{1}{t^2}}} - \sin^{-1} x = -\int \frac{dt}{\sqrt{t^2-1}} - \sin^{-1} x$$

$$= -\log|t + \sqrt{t^2-1}| - \sin^{-1} x + C$$

$$= -\log\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| - \sin^{-1} x + C$$

$$= \log|x| - \log|1 + \sqrt{1-x^2}| - \sin^{-1} x + C$$

17. (b): $f(x) = \tan^{-1} x + \ln\sqrt{1+x} - \ln\sqrt{1-x}$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

$$= \frac{1}{1+x^2} + \frac{1}{1-x^2} = \frac{2}{(1-x^4)}$$

$$\Rightarrow \quad \frac{1}{2} f'(x) = \frac{1}{1-x^4} \Rightarrow \int \frac{1}{2} f'(x)\,dx^4 = \int \frac{dx^4}{1-x^4}$$

$= -\ln(1 - x^4) + c$

18. (d): Integrating by parts, the given integral is equal to

$$x\tan^{-1}\sqrt{\sqrt{x}-1}\Big|_1^{16} - \int_1^{16} \frac{x}{\sqrt{x}} \frac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}} dx$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_1^{16} \frac{dx}{\sqrt{\sqrt{x}-1}}$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_0^{\sqrt{3}} \frac{4t(1+t^2)}{t} dt \qquad (\text{Put } \sqrt{x} = 1 + t^2)$$

$$= \frac{16}{3}\pi - \left(\sqrt{3} + \sqrt{3}\right) = \frac{16}{3}\pi - 2\sqrt{3}$$

19. (c) : Let $g(x) = k^2x^2 + \int_0^x f(t)dt - a$ ...(i)

Since, $a > 0$ and $\int_0^x f(t)dt \to \infty$ as $x \to \pm\infty$ (given)

$$\Rightarrow \quad g(0) = 0 + \int_0^0 f(t)dt - a = -a < 0$$

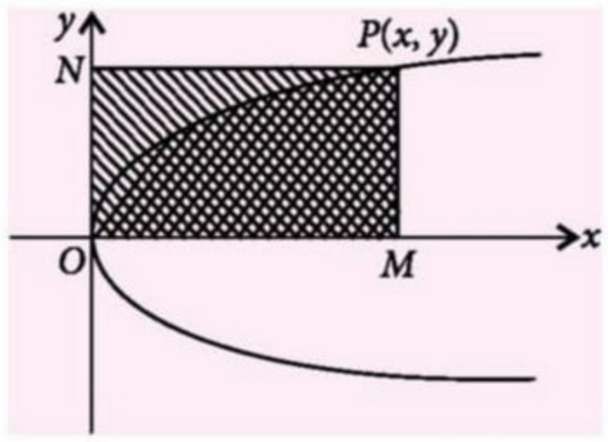
and $g(\infty) = \infty + \infty - a = \infty > 0$

Also, $g(-\infty) = \infty + \infty - a = \infty > 0$

$\Rightarrow$ $g(x)$ is continuous in $(-\infty, \infty)$.

Hence, $g(x) = 0$ has odd number of roots in $(-\infty, 0)$ and odd number of roots in $(0, \infty)$.

20. (b): Let $P(x, y)$ be the point on the curve passing through the origin $O(0, 0)$, and let PN and PM be the lines parallel to the x-axes and y-axes, respectively. If the equation of the curve is $y = y(x)$, the area POM equals $\int_0^x y\,dx$ and the area PON equals $xy - \int_0^x y dx$



Assuming that $2(POM) = PON$, we have

$$2\int_0^x y\,dx = xy - \int_0^x y\,dx \Rightarrow 3\int_0^x y\,dx = xy.$$

Differentiating both sides, we get

$$3y = x\frac{dy}{dx} + y \Rightarrow 2y = x\frac{dy}{dx} \Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

On integrating both sides, we get

$\Rightarrow \log|y| = 2\log|x| + \log C \Rightarrow y = Cx^2$, with C being a constant.

This solutions represents a parabola. We will get a similar result if we had started instead with $2(PON) = POM$.

21. (d): Given curve is

$y = x^4 - 2x^3 + x^2 + 3$

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

$$\frac{d^2y}{dx^2} = 12x^2 - 12x + 2$$

For maxima and minima, put $\frac{dy}{dx} = 0$

$\therefore \quad 4x^3 - 6x^2 + 2x = 0$

$\Rightarrow \quad x = 0, \frac{1}{2}, 1$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{x=0} = 2, \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = -1 \text{ and } \left.\frac{d^2y}{dx^2}\right|_{x=1} = 2$$

$\therefore$ Points of minima are $x = 0$ and $x = 1$

$\therefore$ Required area $= \int_0^1 (x^4 - 2x^3 + x^2 + 3)\,dx$

$$= \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + 3x\right]_0^1$$

$$= \frac{1}{5} - \frac{2}{4} + \frac{1}{3} + 3 = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + 3 = \frac{91}{30} \text{ sq.units}$$

22. (a) : Putting $\frac{dy}{dx} = P$ in given equation, we get

$y = xP + \frac{1}{P}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = P + x\frac{dP}{dx} - \frac{1}{P^2}\frac{dP}{dx}$$

$$\Rightarrow \quad P = P + x\frac{dP}{dx} - \frac{1}{P^2}\frac{dP}{dx} \qquad \left\{\because \ \frac{dy}{dx} = P\right\}$$

$$\Rightarrow \frac{dP}{dx} = 0 \ \text{ or } \ P^2 = \frac{1}{x} \quad \Rightarrow P = C \ \text{ or } \ \left(\frac{dy}{dx}\right)^2 = \frac{1}{x}$$

Put these value in given equation we get $y = Cx + \frac{1}{C}$ which is equation of family of straight lines.

and $y^2 = \left(Px + \frac{1}{P}\right)^2 = P^2x^2 + 2x + \frac{1}{P^2}$

Put $P^2 = \frac{1}{x}$, we get $y^2 = x + 2x + x = 4x$

which represents a parabola.

23. (a) : Here,

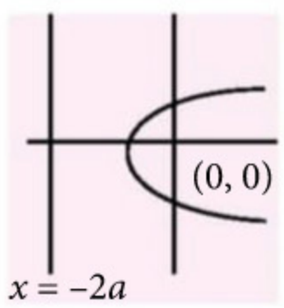
Distance from focus = distance from directrix

$x^2 + y^2 = (2a + x)^2$

$y^2 = 4a(a + x)$ ...(i)

$$2y\frac{dy}{dx} = 4a(0+1)$$

$$\Rightarrow \quad a = \frac{y}{2}\frac{dy}{dx}$$



Using (i), $y^2 = 2y\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx} + x\right)$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

24. (d): $\frac{z_1 - z_2}{z_1 + z_2} = \cos\alpha + i\sin\alpha$

$$\Rightarrow \quad \frac{2z_1}{-2z_2} = \frac{\cos\alpha + i\sin\alpha + 1}{\cos\alpha + i\sin\alpha - 1}$$

$$= \frac{2\cos^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)}{2i\sin(\alpha/2)\cos(\alpha/2) - 2\sin^2(\alpha/2)}$$

$$= \frac{2\cos(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}{2i\sin(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = -\frac{1}{i}\cot\frac{\alpha}{2} \quad \Rightarrow \quad iz_1 = -\cot\frac{\alpha}{2}z_2$$

$\Rightarrow \ iz_1 = Kz_2 \Rightarrow K = -\cot\alpha/2 \Rightarrow \tan\alpha/2 = -1/K$

$$\tan\alpha = \frac{2\tan\alpha/2}{1 - \tan^2\alpha/2} \Rightarrow \frac{-2/K}{1 - 1/K^2} \Rightarrow \frac{-2K}{K^2 - 1}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{2K}{1 - K^2}\right) = 2\tan^{-1}(K)$$

25. (c) : We have $1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{n-1} = 0$.

$$\Rightarrow \quad \sum_{r=0}^{n-1} \alpha^r = 0 \qquad \ldots(i)$$

Now, $\sum_{r=0}^{n-1} |Z_1 + \alpha^r Z_2|^2 = \sum_{r=0}^{n-1} (Z_1 + \alpha^r Z_2)(\bar{Z}_1 + \bar{\alpha}^r \bar{Z}_2)$

$$= \sum_{r=0}^{n-1} |Z_1|^2 + Z_1\bar{Z}_2\sum_{r=0}^{n-1}\bar{\alpha}^r + \bar{Z}_1 Z_2\sum_{r=0}^{n-1}\alpha^r + \sum_{r=0}^{n-1}|Z_2|^2|\alpha|^{2r}$$

$= n(|Z_1|^2 + |Z_2|^2)$, [using (i) and $|\alpha| = 1$.]

26. (b): Let E = Event that A gets six

$\therefore \ P(E) = \frac{1}{6}$

F = Event that B gets six

$\therefore \ P(F) = \frac{1}{6}$

$\therefore \ P(B \text{ wins}) = P(\bar{E}\,F \text{ or } \bar{E}\,\bar{F}\,\bar{E}\,F \text{ or } \bar{E}\,\bar{F}\,\bar{E}\,\bar{F}\,\bar{E}F\ldots)$

(Since B can win the game in 2^{nd}, 4^{th}, 6^{th} …… throw)

$$= \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\frac{1}{6} + \left(\frac{5}{6}\right)^5\frac{1}{6} + \ldots\ldots$$

$$= \frac{5}{36}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots.\right) = \frac{5}{36}\left(\frac{1}{1 - \frac{25}{36}}\right) = \frac{5}{11}$$

27. (d): Let E_1 = the event that six shows when a dice is thrown

E_2 = the number less than or equal to 2 shows when a dice is thrown

$P(E_1) = 1/6$ and $P(E_2) = 2/6 = 1/3$

E = the event that six turns up in the last thrown = the event that E_1 happen in the all previous throws and E_2 happens in the last throw

$P(E) = P(E_1E_2 \text{ or } E_1E_1E_2 \text{ or } E_1E_1E_1E_2\ldots..)$

$= P(E_1)\cdot P(E_2) + P(E_1)\cdot P(E_1)\cdot P(E_2) + P(E_1)\cdot P(E_1)\cdot P(E_1)\cdot P(E_2) + \ldots..$

$$=\frac{1}{3}\cdot\frac{1}{6}+\left(\frac{1}{3}\right)^2\cdot\frac{1}{6}+\left(\frac{1}{3}\right)^3\cdot\frac{1}{6}+.....$$

$$P(E)=\frac{1}{3}\cdot\frac{1}{6}\left[\frac{1}{1-1/3}\right]\Rightarrow P(E)=\frac{1}{12}$$

28. (a) : The given system of equations will be consistent if

$$\Delta=\begin{vmatrix}1 & 1 & -3\\ 1+\lambda & 2+\lambda & -8\\ 1 & -(1+\lambda) & 2+\lambda\end{vmatrix}=0$$

Applying $C_2\to C_2-C_1$ and $C_3\to C_3+3C_1$, we get

$$\Delta=\begin{vmatrix}1 & 0 & 0\\ 1+\lambda & 1 & -5+3\lambda\\ 1 & -2-\lambda & 5+\lambda\end{vmatrix}=0$$

$\Rightarrow$ $(5+\lambda)+(2+\lambda)(3\lambda-5)=0$

$\Rightarrow$ $5+\lambda+6\lambda-10+3\lambda^2-5\lambda=0$

$\Rightarrow$ $3\lambda^2+2\lambda-5=0\Rightarrow(3\lambda+5)(\lambda-1)=0$

$\Rightarrow$ $\lambda=-5/3$ or $\lambda=1$.

29. (b): For the system of equations having no solution, we must have

$$\frac{k+1}{k}=\frac{8}{k+3}\neq\frac{4k}{3k-1}$$

$\Rightarrow$ $(k+1)(k+3)=8k$ and $8(3k-1)\neq 4k(k+3)$

But $(k+1)(k+3)=8k\Rightarrow k^2+4k+3=8k$

or $k^2-4k+3=0$ or $(k-1)(k-3)=0\Rightarrow k=1$ or 3

For $k=1$, $8(3k-1)=16$ and $4k(k+3)=16$

$\therefore$ $8(3k-1)=4k(k+3)$ for $k=1$

For $k=3$, $8(3k-1)=64$ and $4k(k+3)=72$

i.e. $8(3k-1)\neq 4k(k+3)$ for $k=3$.

Thus, there is just one value for which the given system of equations has no solution.

30. (b): $f(A)=I+A+A^2+.....+A^{16}$

Since, $A=\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}\Rightarrow A^2=\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$A^3=A^2\cdot A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Similarly, $A^4=A^5=.....=A^{16}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$f(A)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+....+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1 & 5\\ 0 & 1\end{bmatrix}$$

31. (c) : $\vec{a}\perp(\vec{b}+\vec{c})\Rightarrow\vec{a}\cdot(\vec{b}+\vec{c})=0$

$\Rightarrow$ $\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$

Also, $\vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a}=0$ and $\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=0$

Adding, $2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$=9+16+25+0=50$

$\therefore$ $|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$

32. (b): Lines are coplanar if $\begin{vmatrix}1 & 1 & 1\\ 2 & x_1 & x_2\\ 3 & 4 & 5\end{vmatrix}=0$

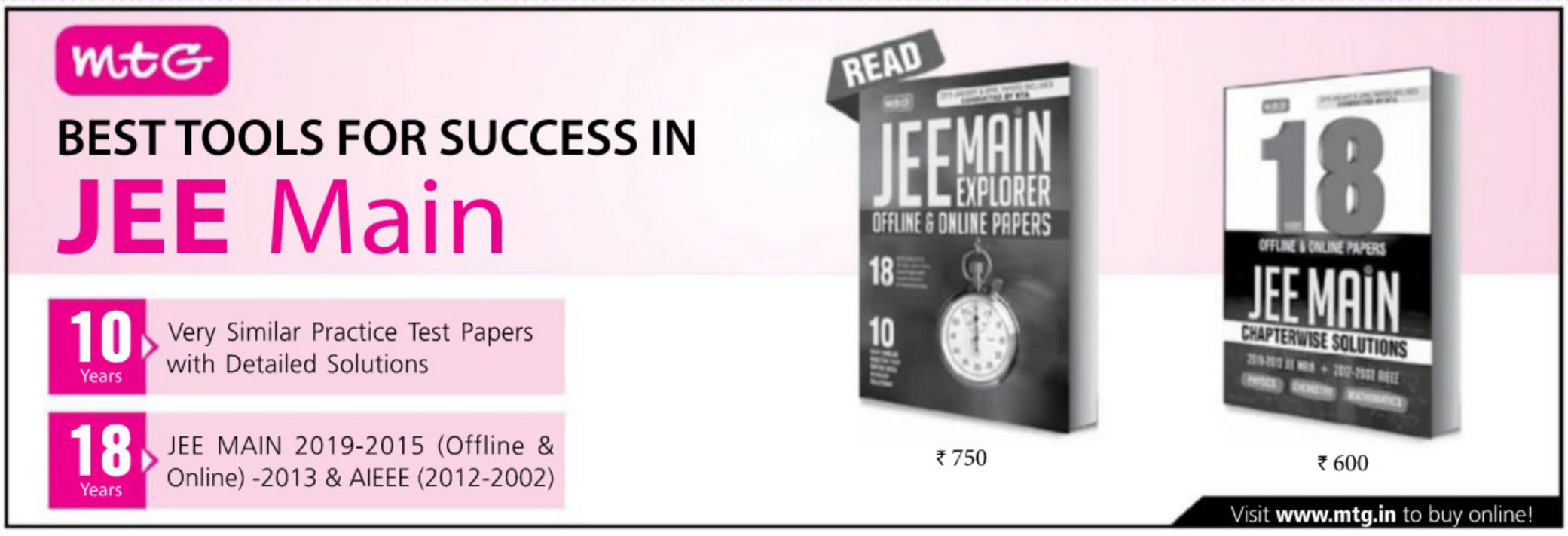
$\Rightarrow$ $2x_1-x_2=2$

Also, sum of roots $=\dfrac{-(x_2+2)}{x_1}=\dfrac{-2x_1}{x_1}=-2$

33. (a) : Variance $=(\text{S.D.})^2=\dfrac{1}{n}\Sigma x^2-\left(\dfrac{\Sigma x}{n}\right)^2$,

$\left(\because\ \bar{x}=\dfrac{\Sigma x}{n}\right)$

$$=\frac{n(n+1)(2n+1)}{6n}-\left(\frac{n(n+1)}{2n}\right)^2=\frac{n^2-1}{12}.$$



QUANTITATIVE APTITUDE

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1. A young girl counted in the following way on the fingers of her left hand. She started calling the thumb 1, the index finger 2, middle finger 3, ring finger 4, little finger 5, then reversed direction, calling the ring finger 6, middle finger 7, index finger 8, thumb 9 and then back to the index finger for 10, middle finger for 11, and so on. She counted up to 1994. She ended on her

(a) Thumb (b) Index finger
(c) Middle finger (d) Ring finger

2. *A*, *B*, *C*, *D*, ..., *X*, *Y*, *Z* are the players who participated in a tournament. Everyone played with every other player exactly once. A win scores 2 points, a draw scores 1 point and a lose scores 0 point. None of the matches ended in a draw. No two players scored the same score. At the end of the tournament, a ranking list is published which is in accordance with the alphabetical order. Then :

(a) *M* wins over *N* (b) *N* wins over *M*
(c) *E* wins over *D* (d) None of these

3. The following table presents the sweetness of different items relative to sucrose, whose sweetness is taken to be 1.00.

Lactose	0.16
Maltose	0.32
Glucose	0.74
Sucrose	1.00
Fructose	1.70
Saccharin	675.00

Approximately how many times sweeter than sucrose is a mixture consisting of glucose; sucrose and fructose in the ratio of 1 : 2 : 3?

(a) 2.3 (b) 1.0
(c) 0.3 (d) None of these

4. Anjali bought some chocolates from Nestle's exclusive shop, she gave to Amit one less than half of what she had initially. Then she had given 3 chocolates to Bablu and then half of the chocolates which she had then gave to Charles. Thus finally she gave one chocolate to Deepak and the remaining one she ate herself. The number of chocolates she had purchased.

(a) 9 (b) 12 (c) 10 (d) 15

5. A water tank has three taps : *A*, *B* and *C*. *A* fills 4 buckets in 24 mins, *B* fills 8 buckets in 1 hr and *C* fills 2 buckets in 20 mins. If all the taps are opened together, a full tank is emptied in 2 hrs. If a bucket contains 5 L water, what is the capacity of the tank?

(a) 120 L (b) 240 L (c) 180 L (d) 60 L

6. The number of votes not cast for the Praja Party increased by 25% in the National General Election over those not cast for it in the previous Assembly Polls, and the Praja Party lost by a majority twice as large as that by which it had won the Assembly Polls. If a total 2,60,000 people voted each time, how many voted for the Praja Party in the previous Assembly Polls?

(a) 1,10,000 (b) 1,50,000
(c) 1,40,000 (d) 1,20,000

7. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of the three popular options-air conditioning, radio and power windows-were already installed. The survey found : 15 had air conditioning, 2 had air conditioning and power windows but no radios, 12 had radio, 6 had air conditioning and radio but no power windows, 11 had power windows, 4 had radio and power windows, 3 had all three options. What is the number of cars that had none of the options?

(a) 4 (b) 3 (c) 1 (d) 2

8. A farmer has decided to build a wire fence along one straight side of his property. For this, he planned to place several fence-posts at 6 m intervals, with posts fixed at both ends of the side. After he bought the posts and wire, he found that the number of posts he had bought was 5 less than required. However, he discovered that the number of posts he had bought would be just sufficient if he spaced them 8 m apart. What is the length of the side of his property and how many posts did he buy?

(a) 100 m, 15 (b) 100 m, 16
(c) 120 m, 15 (d) 120 m, 16

9. Consider two different cloth-cutting processes. In the first one, n circular cloth pieces are cut from a square cloth piece of side a in the following steps : the original square of side a is divided into n smaller squares, not necessarily of the same size; then a circle of maximum possible area is cut from each of the smaller squares. In the second process, only one circle of maximum possible area is cut from the square of side a and the process ends there. The cloth pieces remaining after cutting the circles are scrapped in both the processes. The ratio of the total area of scrap cloth generated in the former to that in the latter is

(a) 1 : 1 (b) $\sqrt{2}:1$
(c) $\frac{n(4-\pi)}{4n-\pi}$ (d) $\frac{4n-\pi}{n(4-\pi)}$

10. Three bells chime at intervals of 18 mins, 24 mins and 32 mins respectively. At a certain time, they begin to chime together. What length of time will elapse before they chime together again?

(a) 2 hrs and 24 mins (b) 4 hrs and 48 mins
(c) 1 hrs and 36 mins (d) 5 hrs

11. I started climbing up the hill at 6 a.m. and reached the top of the temple at 6 p.m. Next day, I started coming down at 6 a.m. and reached the foothill at 6 p.m. I walked on the same road. The road is so short that only one person can walk on it. Although given I varied my pace on my way, I never stopped on my way. Then which of the following must be true?

(a) My average speed downhill was greater than that of uphill
(b) At noon, I was at the same spot on both the days.
(c) My average speed uphill was greater than downhill.
(d) None of these

12. A rectangle $PRSU$, is divided into two smaller rectangles $PQTU$, and $QRST$ by the line $TQ.PQ = 10$ cm. $QR = 5$ cm and $RS = 10$ cm. Points A, B, F are within rectangle $PQTU$, and points C, D, E are within the rectangle $QRST$. The closest pair of points among the pairs (A, C), (A, D), (A, E), (F, C), (F, D), (F, E), (B, C), (B, D), (B, E) are $10\sqrt{3}$ cm apart.

Which of the following statements is necessarily true?

(a) The closest pair of points among the six given points cannot be (F, C).
(b) Distance between A and B is greater than that between F and C.
(c) The closest pair of points among the six given points is (C, D), (D, E) or (C, E).
(d) Cannot be determined

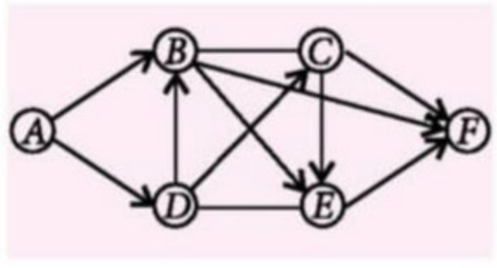
13. A watch dealer incurs an expense of ₹ 150 for producing every watch. He also incurs an additional expenditure of ₹ 30,000, which is independent of the number of watch produced. If he is able to sell a watch during the season, he sells it for ₹ 250. If he fails to do so he has to sell each watch for ₹ 100. If he is able to sell only 1200 out of 1500 watches he has made in the season, then he has made a profit of :

(a) ₹ 90,000 (b) ₹ 75,000
(c) ₹ 45,000 (d) ₹ 60,000

14. A child was asked to add first few natural numbers (that is, 1 + 2 + 3 + ...) so long his patience permitted. As he stopped he gave the sum as 575. When the teacher declared the result wrong the child discovered he had missed one number in the sequence during addition. The number he missed was

(a) Less than 10 (b) 10
(c) 15 (d) More than 15

15. The given figure shows the network connecting cities A, B, C, D, E and F. The arrows indicate permissible direction of travel. What is the number of distinct path from A to F?



(a) 9 (b) 10
(c) 11 (d) None of these

16. An intelligence agency forms a code of two distinct digits selected from 0, 1, 2, ..., 9 such that the first digit of the code is nonzero. The code, handwritten on a slip, can however potentially create confusion, when read upside down-for example, the code 91 may appear as 16. How many codes are there for which no such confusion can arise?

(a) 80 (b) 78
(c) 71 (d) 69

17. Read the information given below and answer the question that follows:

The batting average (BA) of a test batsman is computed from runs scored and innings played-completed innings and incomplete innings (not out) in the following manner

r_1 = number of runs scored in completed innings

n_1 = number of completed innings

r_2 = number of runs scored in incomplete innings

n_2 = number of incomplete innings

$$BA = \frac{r_1 + r_2}{n_1}$$

To better assess a batsman's accomplishments, the ICC is considering two other measures $MB\ A_1$ and $MB\ A_2$ defined as follows :

$$MB\ A_1 = \frac{r_1}{n_1} + \frac{r_2}{n_2}\max\left[0, \left(\frac{r_1}{n_2} - \frac{r_1}{n_1}\right)\right] : MB\ A_2 = \frac{r_1 + r_2}{n_1 + n_2}$$

An experienced cricketer with no incomplete innings has a BA of 50. The next time he bats, the innings is incomplete and he scores 45 runs. It can be inferred that

(a) BA and $MB\ A_1$ both will decrease.
(b) BA will increase and $MB\ A_2$ will decrease.
(c) BA will increase and not enough data is available to assess change in $MB\ A_1$ and $MB\ A_2$.
(d) None of these

18. An airline has a certain free luggage allowance and charges for excess luggage at a fixed rate per kg. Two passengers, Ananya and Krishi have 60 kg of luggage between them, and are charged ₹ 1200 and ₹ 2400, respectively for excess luggage. Had the entire luggage belonged to one of them, the excess luggage charge would have been ₹ 5400.
What is the weight of Krishi's luggage?

(a) 20 kg (b) 25 kg
(c) 30 kg (d) 35 kg

Directions (19-20) : Answer the questions based upon following information :
Each of the question is followed by two statements, I and II.

Choose 1. If the question can be answered with the help of statement I alone.

Choose 2. If the question can be answered with the help of statement II alone.

Choose 3. If the question can be answered with the help of both statements together.

Choose 4. If the question cannot be answered even with the help of both statements together.

19. After what time will the two persons Tez and Gati meet while moving around the circular track? Both of them start at the same point and at the same time.

I. Tez moves at a constant speed of 5 m/s, while Gati starts at a speed of 2 m/s and increased his speed by 0.5 m/s at the end of every second thereafter.
II. Gati can complete one entire lap in exactly 10 s.

(a) 1 (b) 2 (c) 3 (d) 4

20. What is the area bounded by the two lines and the coordinate axes in the first quadrant?

I. The lines intersect at a point which also lies on the lines $3x - 4y = 1$ and $7x - 8y = 5$.
II. The lines are perpendicular, and one of them intersects the Y-axis at an intercept of 4.

(a) 1 (b) 2 (c) 3 (d) 4

21. Mr. Bankatlal acted as a judge for the beauty contest. There were four participants, viz. Ms. Andhra Pradesh, Ms. Uttar Pradesh, Ms. West Bengal and Ms. Maharashtra, Mrs. Bankatlal, who was very anxious about the result asked him about it as soon as he was back home. Mr. Bankatlal just told that the one who was wearing the yellow saree won the contest. When Mrs. Bankatlal pressed for further details, he elaborated as follows:

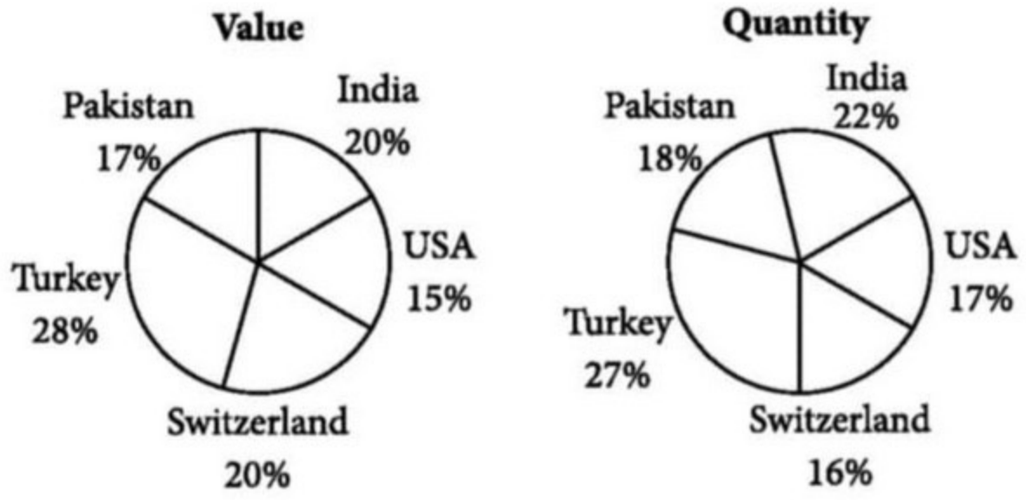
Monthly Test Drive-2 CLASS XII ANSWER KEY

1. (c)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (c)
6. (c)	**7.** (a,b)	**8.** (b,c)	**9.** (b,c)	**10.** (b)
11. (a,b)	**12.** (a)	**13.** (a,b,d)	**14.** (d)	**15.** (b)
16. (d)	**17.** (2)	**18.** (3)	**19.** (2)	**20.** (7)

A. All of them were sitting in a row.
B. All of them wore sarees of different colours, *viz.* green, yellow, white, red.
C. There was only one runner-up and she was sitting beside Ms. Maharashtra.
D. The runner-up was wearing the green saree.
E. Ms. West Bengal was not sitting at the ends and was not the runner up.
F. Ms. Maharashtra was wearing white saree.
G. Ms. Andhra Pradesh was not wearing the green saree.
H. Participants wearing yellow saree and white saree were at the ends.

Who wore the red saree?
(a) Ms. Andhra Pradesh
(b) Ms. West Bengal
(c) Ms. Uttar Pradesh
(d) Ms. Maharashtra

Directions (22-23) : The pie charts give the data about a textile manufacturing unit exporting to different countries.



The total value is 5760 million rupees and the total quantity is 1.055 million tonnes.

22. Which country has the highest price for its supply?
(a) Pakistan (b) Turkey
(c) Switzerland (d) India

23. What is the price in ₹ per kg for Turkey?
(a) 6.3 (b) 5.6
(c) 4.8 (d) 4.5

24. A, B and C are defined as follows :

$A = (2.000004) \div [(2.000004)^2 + (4.000008)]$,

$B = (3.000003) \div [(3.000003)^2 + (9.000009)]$,

$C = (4.000002) \div [(4.000002)^2 + (8.000004)]$,

Which of the following is true about the values of the above three expression?
(a) All of them lie between 0.18 and 0.2
(b) A is twice of C
(c) C is the smallest
(d) B is the smallest

25. You can collect rubies and emeralds as many you can. Each ruby is worth ₹ 4 crores and each emerald is wroth of ₹ 5 crore. Each ruby weights 0.3 kg and each emerald weighs 0.4 kg. Your bag can carry at the most 12 kg. What you should collect to get the maximum wealth?
(a) 20 rubies and 15 emeralds
(b) 40 rubies
(c) 28 rubies and 9 emeralds
(d) None of these

ANSWER KEYS

1.	(b)	**2.**	(a)	**3.**	(d)	**4.**	(b)	**5.**	(b)
6.	(c)	**7.**	(d)	**8.**	(d)	**9.**	(a)	**10.**	(b)
11.	(d)	**12.**	(d)	**13.**	(b)	**14.**	(d)	**15.**	(d)
16.	(d)	**17.**	(b)	**18.**	(d)	**19.**	(d)	**20.**	(c)
21.	(b)	**22.**	(c)	**23.**	(b)	**24.**	(d)	**25.**	(b)

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